Sublogarithmic uniform Boolean Proof Nets

DICE 2011 Second Workshop on Developments in Implicit Computational Complexity

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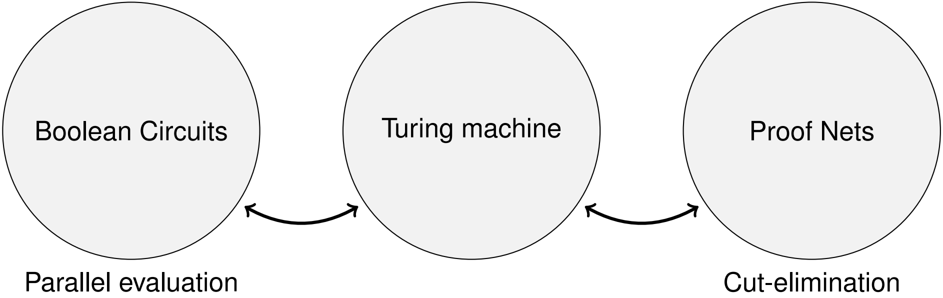
2nd-3rd April 2011, Saarbrücken

Two models of parallel computation



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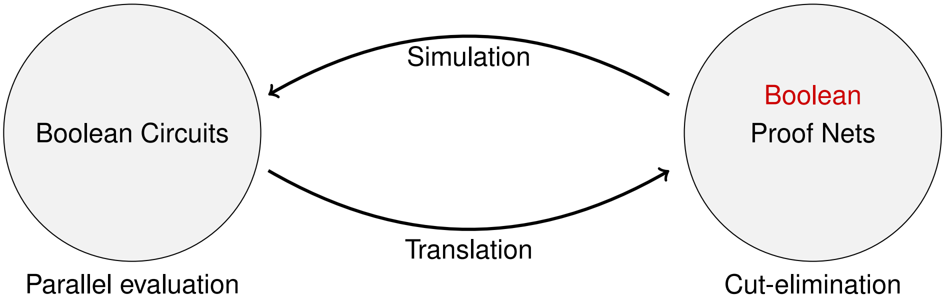

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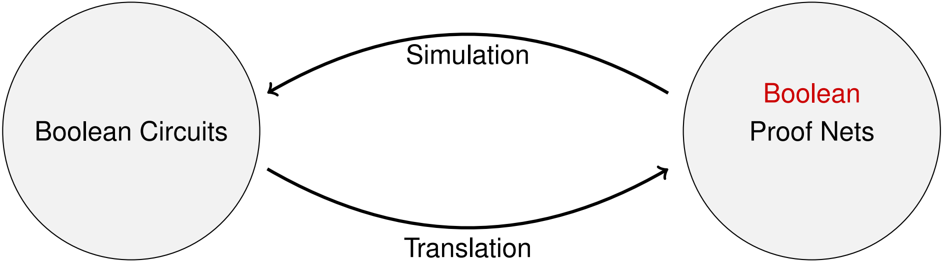

Parallel evaluation

Cut-elimination

Implicit complexity framework:

- Size and depth
- Resources needed by the reductions
- Uniformity

Previous works and new results

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- [Terui, 2004] did not took into account uniformity.
- [Mogbil and Rahli, 2007] did, but translation was in L.

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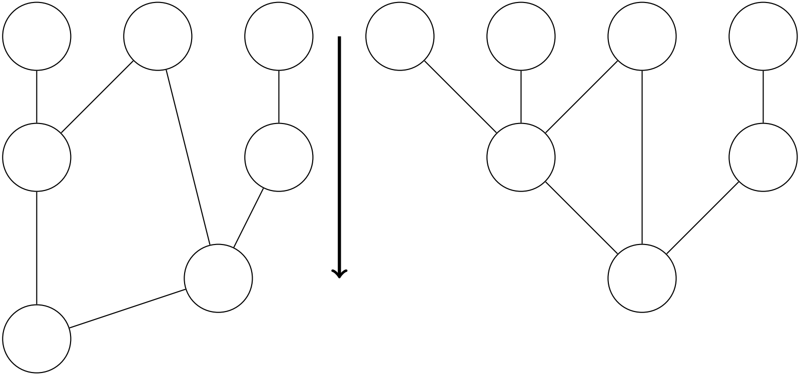
Results

- *Proof Circuits lighten the size and translate all kind of Boolean Circuits.*
- *Translation in AC^0 concerning uniform families.*
- *Proof Nets compute functions under L.*

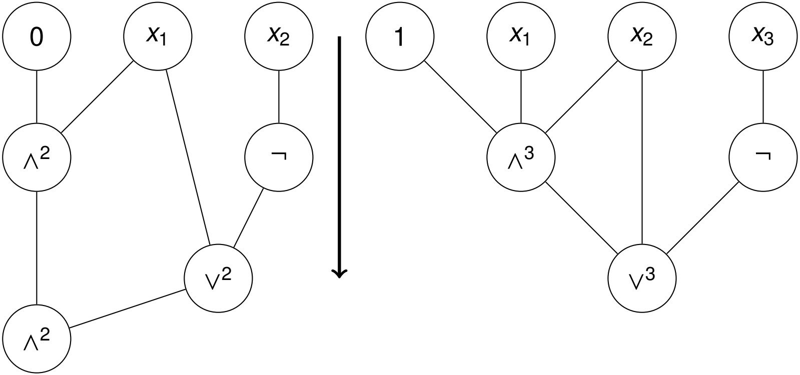
Map

- First definitions: Boolean Circuits and Boolean Proof Nets
- Simulation of Boolean Proof Nets by Boolean Circuits
- Examples of translation and composition
- Proof Circuits and translation

Examples (Two Boolean Circuits)



Examples (Two Boolean Circuits)



Definition (Basis)

$\mathfrak{B}_0 = \{\neg, \wedge^2, \vee^2\}$
$\mathfrak{B}_1 = \{\neg, (\wedge^n)_{n \geqslant 2}, (\vee^n)_{n \geqslant 2}\}.$

Definition (NC^i resp. AC^i)

For $i \in \mathbb{N}$, set of boolean functions computable by a uniform family of Boolean Circuits $C = (C_n)$ where for all C_n

- *its depth is $O(\log^i n)$,*
- *its size is polynomial in n,*
- *its gate are labeled with functions of $\mathfrak{B}_0$ resp. $\mathfrak{B}_1$.*

$$NC = \bigcup_{i \in \mathbb{N}} NC^i = \bigcup_{i \in \mathbb{N}} AC^i = AC$$

Complexity Classes: *NC* and *AC*

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Theorems

$$AC^0 \subsetneq NC^1 \subseteq L \subseteq NL \subseteq AC^1 \subseteq NC^1 \subseteq AC^2 \subseteq \ldots$$

$$AC^0(UstCONN_2) \subseteq AC^1 \supseteq L$$

Definition (Rules of **MLLu**)

$$\frac{}{\vdash A, A^\perp}\, ax. \qquad \frac{\vdash \Gamma_1, A_1 \quad \dots \quad \vdash \Gamma_n, A_n}{\vdash \Gamma_1, \dots, \Gamma_n, \otimes^n(A_1, \dots, A_n)}\, \otimes^n$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}\, cut \qquad \frac{\vdash \Gamma, A_n, \dots, A_1}{\vdash \Gamma, \invamp^n(A_n, \dots, A_1)}\, \invamp^n$$

Definition (Boolean Type, [Terui, 2004])

$$\dfrac{\dfrac{\dfrac{}{\vdash \alpha^\perp, \alpha}\, ax. \qquad \dfrac{}{\vdash \alpha^\perp, \alpha}\, ax.}{\vdash \alpha^\perp, \alpha^\perp, \alpha \otimes \alpha}\, \otimes^2}{\vdash \invamp^3(\alpha^\perp, \alpha^\perp, \alpha \otimes \alpha)}\, \invamp^3$$

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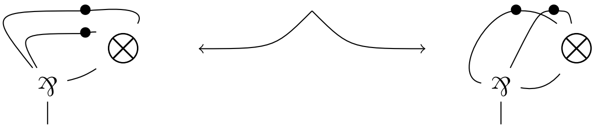

MLLu

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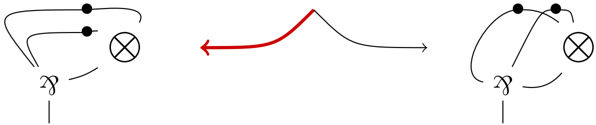
Definition (Boolean Type, [Terui, 2004])

$$\dfrac{\dfrac{\dfrac{}{\vdash p : \alpha^{\perp}, p : \alpha \rhd ax_p}\ ax. \qquad \dfrac{}{\vdash q : \alpha^{\perp}, q : \alpha \rhd ax_q}\ ax.}{\vdash p : \alpha^{\perp}, q : \alpha^{\perp}, r : \alpha \otimes \alpha \rhd tensor_r^{p,q}(ax_p, ax_q)}\ \otimes^2}{\vdash s : \parr^3(\alpha^{\perp}, \alpha^{\perp}, \alpha \otimes \alpha) \rhd par_s^{q,p,r}(tensor_r^{p,q}(ax_p, ax_q))}\ \parr^3$$

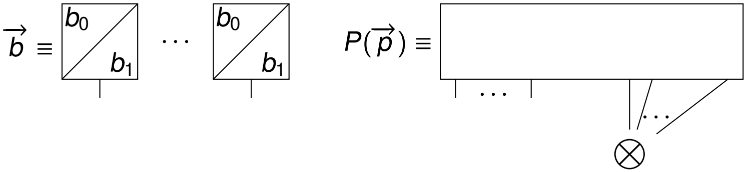


Definition (Boolean Proof Nets, [Terui, 2004])

A Boolean Proof Net $P(\overrightarrow{p})$ with n inputs is a Proof Net of type

$$\vdash p_1 : \boldsymbol{B}^{\perp}[A_1], \ldots, p_n : \boldsymbol{B}^{\perp}[A_n], s : \otimes^{1+m}(\boldsymbol{B}[A], C_1, \ldots, C_m)$$

Definition ($P(\overrightarrow{b})$)



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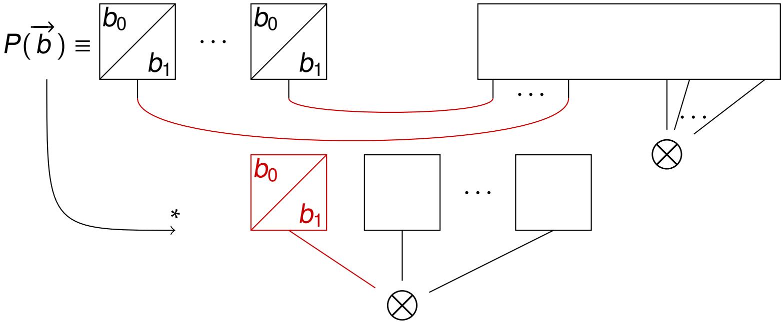
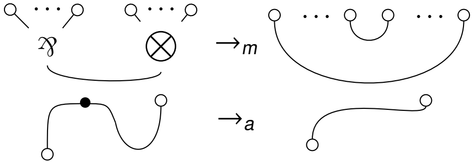
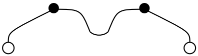


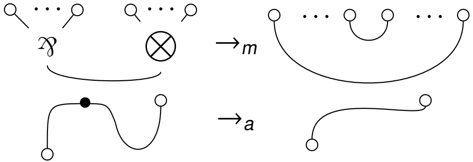
Definition ($\rightarrow_m$ and $\rightarrow_a$)



Remark (Critical pair)



Definition ($\to_m$ and $\to_a$)



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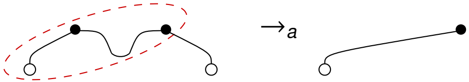

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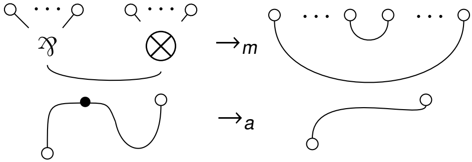

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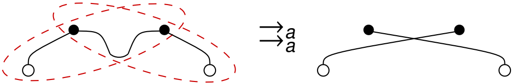

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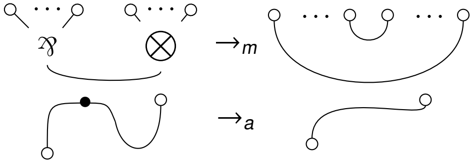

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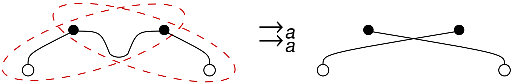
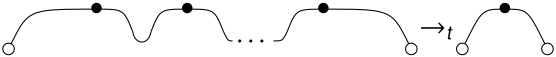


Definition (t-reduction)



Map

- First definitions: Boolean Circuits and Boolean Proof Nets
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Simulation of $\rightrightarrows$ by Boolean Circuits

Theorem ([Terui, 2004])

Every Proof-Net P normalizes in at most $3 \times d(P)$ steps of parallel reduction ($\rightrightarrows_t$, $\rightrightarrows_a$, $\rightrightarrows_m$).

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Description of a Boolean Proof Net P_n

$\downarrow$

Simulates $\rightrightarrows_t$

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$$\left.\begin{array}{c} \boxed{\text{Simulates } \rightrightarrows_t} \\ \boxed{\text{Simulates } \rightrightarrows_a} \\ \boxed{\text{Simulates } \rightrightarrows_m} \end{array}\right\} d(P_n) \text{ times}$$

$\vdots$

Establishes the output of P_n

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Constant-depth size $O(|P_n|^3)$ $\left\{ \begin{array}{c} \boxed{\text{Simulates } \rightrightarrows_t} \\ \boxed{\text{Simulates } \rightrightarrows_a} \\ \boxed{\text{Simulates } \rightrightarrows_m} \end{array} \right\}$ $d(P_n)$ times

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Description of a Boolean Proof Net P_n

$\downarrow$

Constant-depth size $O(|P_n|^3)$

Simulates $\rightrightarrows_t$

Simulates $\rightrightarrows_a$

Simulates $\rightrightarrows_m$

$d(P_n)$ times

$\vdots$

Establishes the output of P_n

Depth $d(P_n)$ size $O(|P_n|^4)$ over $\mathfrak{B}_1(UstCONN_2)$

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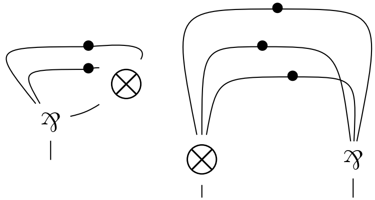
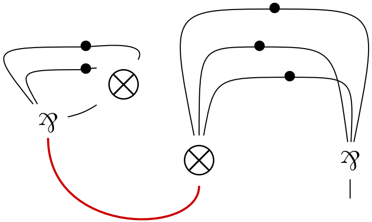
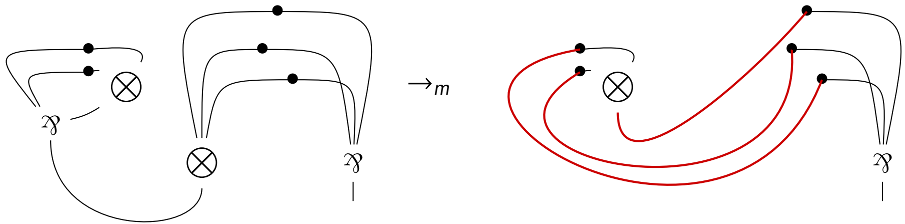


First example: negation




⅋
⅋
$\to_m$
⅋

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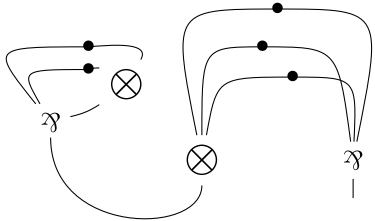
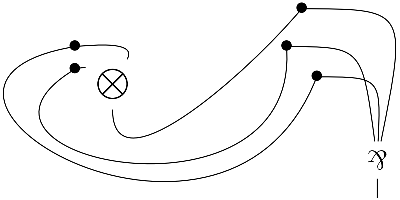
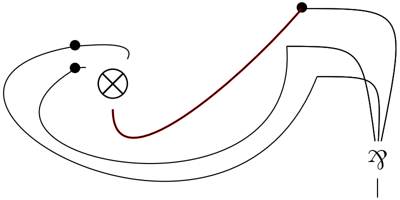


$\rightarrow_m$



$\downdownarrows_t$



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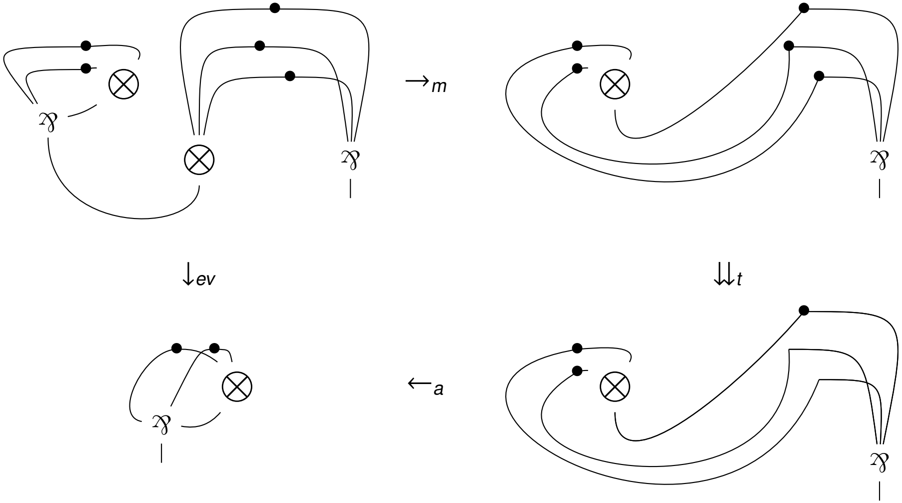
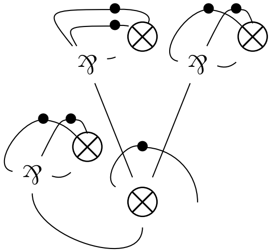


Second example: conditional



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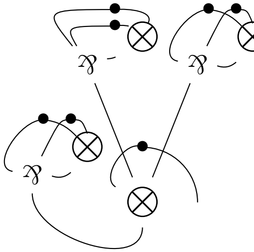
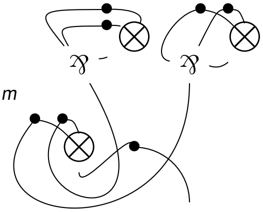
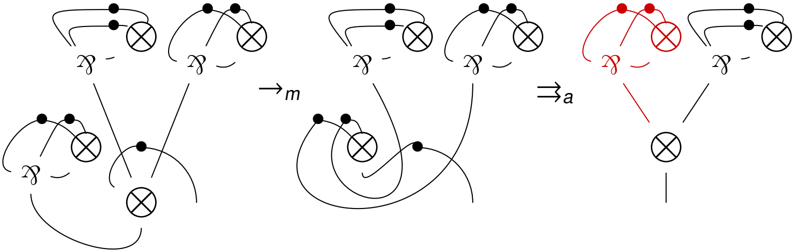
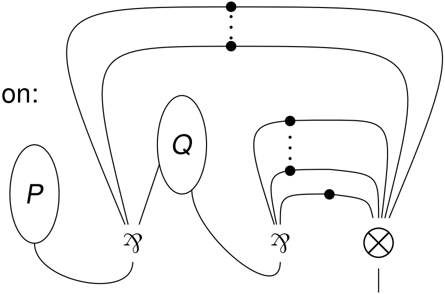


$\rightarrow_m$

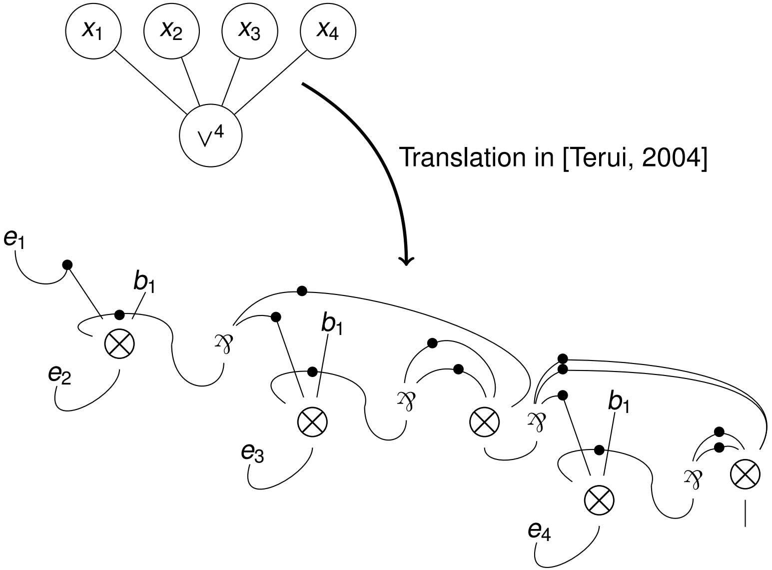




Composition:



$\vee^4$ translated



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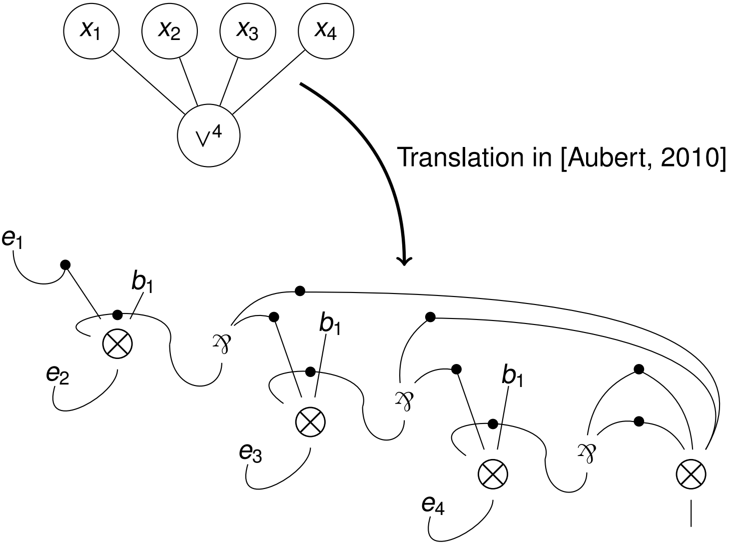

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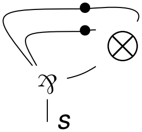
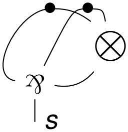
Pieces (1/4)

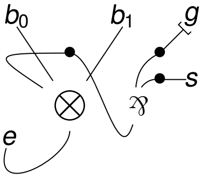
$b_0 \equiv$



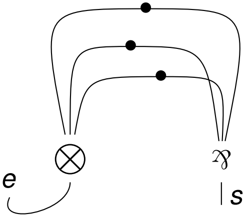
$b_1 \equiv$



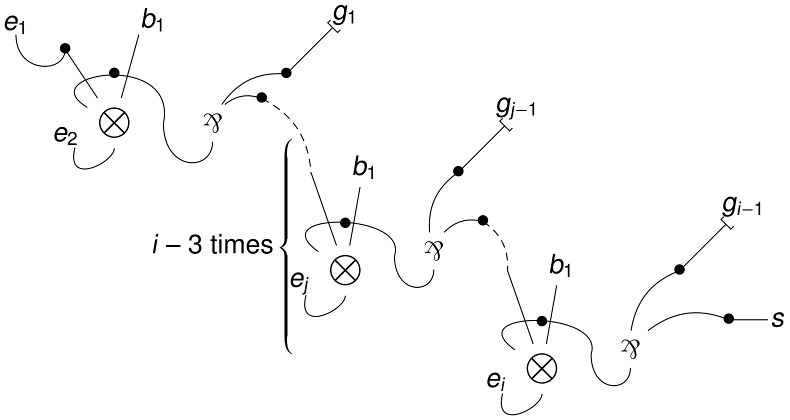
$COND \equiv$



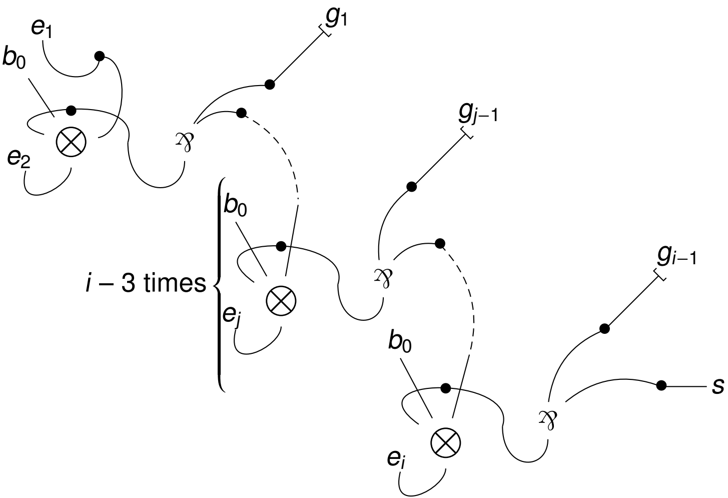
$NEG \equiv$



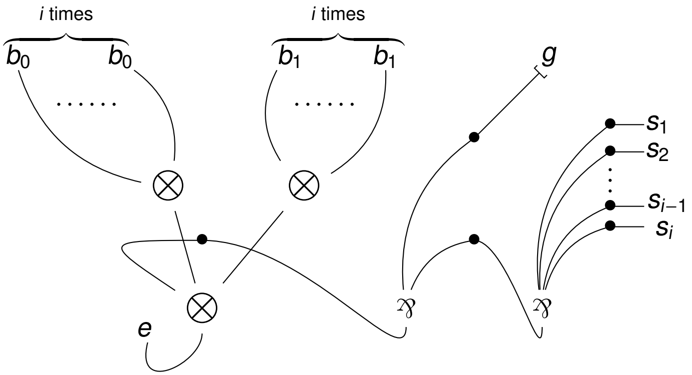
$DISJ^i \equiv$



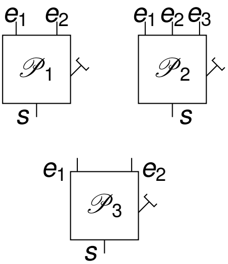
$CONJ^i \equiv$



$DUPL^i \equiv$



Definition (Proof Circuits)



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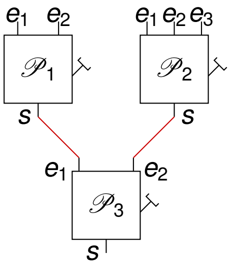

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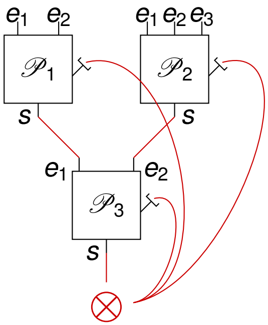

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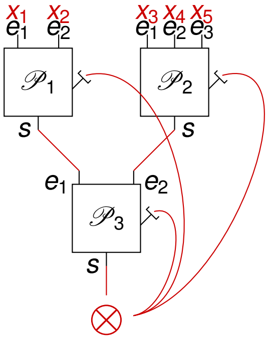

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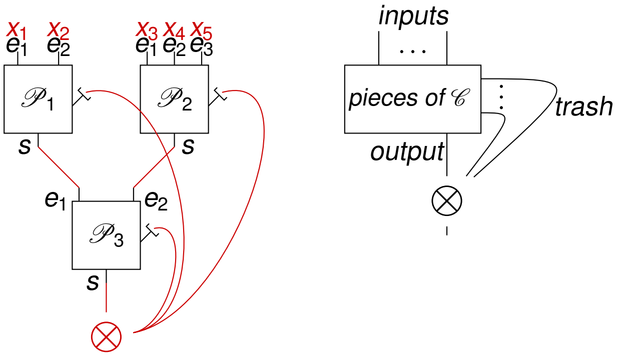

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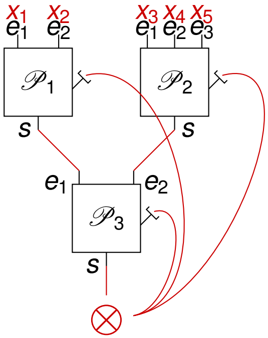

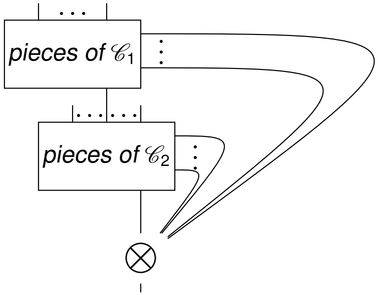

Theorem ([Aubert, 2010])

Every Proof Circuits is a Boolean Proof Net.

Complexity Class: *CCP*

Definition (*CCP*^{i} [Aubert, 2010])

For $i \in \mathbb{N}$, set of boolean functions computable by a uniform family of Proof Circuits $P = (P_n)$ where for all P_n

- *its depth is $O(\log^i n)$,*
- *its size is polynomial in n.*

$$CCP = \bigcup_{i \in \mathbb{N}} CCP^i$$

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For $i \in \mathbb{N}$, set of boolean functions computable by a uniform family of Proof Circuits $P = (P_n)$ where for all P_n

- *its depth is $O(\log^i n)$,*
- *its size is polynomial in n.*

$$CCP = \bigcup_{i \in \mathbb{N}} CCP^i$$

Theorem (Simulation)

For all $i \in \mathbb{N}$,

$$CCP^i \subseteq AC^i(UstCONN_2)$$

Definition (Translation from AC^i to CCP^i)

Input: *Description of a uniform familly of Boolean Circuits* $C = (C_n)$ *in* AC^i.

Output: *Description of a familly of Proof Circuits* $P = (P_n)$ *in* CCP^i *so that for all* k, *for all* $\overrightarrow{b}$, $P_k(\overrightarrow{b}) \rightarrow_{ev} b_j$ *iff* $C_k(\overrightarrow{b})$ *evaluates to* j.

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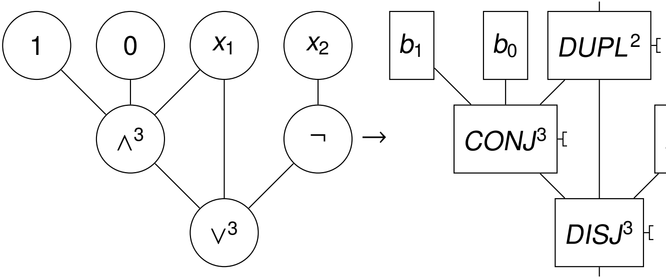

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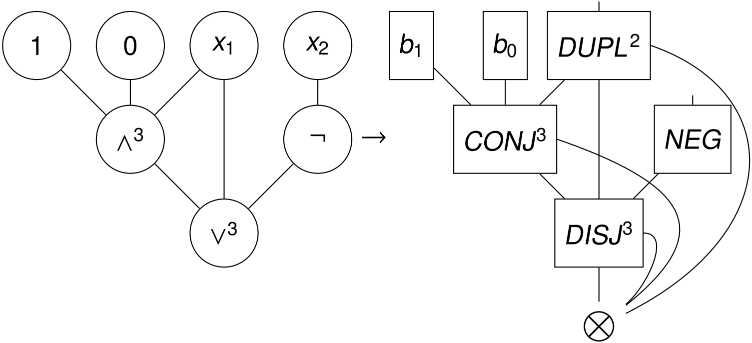

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Corollaries

- *Uniformity is preserved*
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- *For all* $i \in \mathbb{N}$, $AC^i \subseteq PCC^i \subseteq AC^i(UstConn_2)$

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Future work:

- $UstConn_2 \in L$, is there a lighter function?
- Link between Proof Nets and Alternating Turing Machines?

Bibliography... and thanks

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Thanks!