Characterizing **co-NL** by a Group Action

Séminaire L.I.P.N.

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$$\textbf{co-NL} \quad = \quad \{ANDPM\} = \{NPPM\} \quad = \quad \{P_+\} = \{P_{\leqslant 0}\} \quad = \quad \textbf{co-NL}$$

(A)NDPM Observations

co-NL $\subseteq$ $\{ANDPM\} \subseteq \{NPPM\}$ $\subseteq$ $\{P_+\} \subseteq \{P_{\leqslant 0}\}$ $\subseteq$ **co-NL**

(A)NDPM Observations

$$(\mathfrak{M}_6(\mathfrak{S}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C})\,)\,(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

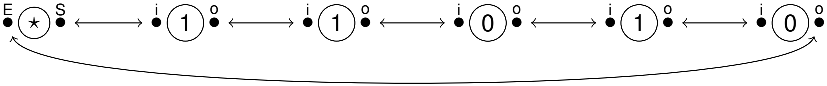
$$\underbrace{(\mathfrak{M}_6(\mathfrak{S}) \otimes \overbrace{\mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C})}^{=\mathfrak{O}})}_{\text{Observation}} \underbrace{(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})}_{\text{The input}}$$

Definition (Observations)

Let $(\mathfrak{N}_0, \mathfrak{S})$ be a normative pair. An *observation* is an operator $\phi \in \mathfrak{M}_6(\mathfrak{S}) \otimes \mathfrak{O}$, where $\mathfrak{O}$ is a matrix algebra, i.e. $\mathfrak{O} = \mathfrak{M}_d(\mathbb{C})$ for $d \in \mathbb{N}$, called the *algebra of states*.

Definition (Binary representation of integers)

An operator $N_n \in \mathfrak{M}_6(\mathfrak{N}_0)$ is a *binary representation* of an integer n if . . .



Our framework: an algebra

$$\underbrace{\left(\mathfrak{M}_6(\mathfrak{S}) \otimes \overbrace{\mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C})}^{=\mathfrak{O}}\right)}_{\text{Observation}} \underbrace{\left(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O}\right)}_{\text{The input}}$$

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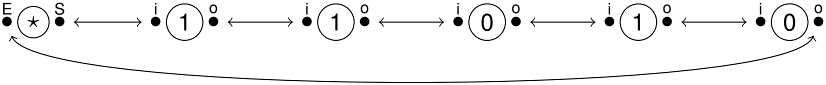

The computation ends if $\exists k \in \mathbb{N}$ such that

$$(\phi(N_n \otimes 1_\circ))^k = 0$$

$$(\mathfrak{M}_6(\mathfrak{S}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

Definition (Normative Pairs)

Let $\mathfrak{N}_0$ and $\mathfrak{S}$ be two subalgebras of a von Neumann algebra $\mathfrak{M}$. The pair $(\mathfrak{N}_0, \mathfrak{S})$ is a *normative pair (in $\mathfrak{M}$)* if:

- $\mathfrak{N}_0$ is isomorphic to $\mathfrak{R}$;
- For all $\Phi \in \mathfrak{M}_6(\mathfrak{S})$ and $N_n, N'_n \in \mathfrak{M}_6(\mathfrak{N}_0)$ two binary representations of n,

$$\Phi N_n \text{ is nilpotent} \Leftrightarrow \Phi N'_n \text{ is nilpotent}$$

$$(\mathfrak{M}_6(\mathfrak{S}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

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Proposition

Let G be the group of finite permutations over $\mathbb{N}$, α an action of G and for all $n \in \mathbb{N}$, $\mathfrak{N}_n = \mathfrak{R}$. The algebra$(\otimes_{n\in\mathbb{N}}\mathfrak{N}_s) \rtimes_{\hat{\alpha}} G$ contains a subalgebra generated by G that we will denote $\mathfrak{G}$.
$(\mathfrak{N}_0, \mathfrak{G})$ is a normative pair in $(\otimes_{n\in\mathbb{N}}\mathfrak{N}_s) \rtimes_{\hat{\alpha}} G$ (the type II_1 hyperfinite factor).

$$(\mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

Definition ($P_{\geqslant 0}$ and P_+)

Let $(\mathfrak{N}_0, \mathfrak{G})$ be a normative pair, $(\phi_{i,j})_{0 \leqslant i,j \leqslant 6d} \in \mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_d(\mathbb{C})$ an observation , we define:

$$[\phi] = \{n \in \mathbb{N} \mid \phi(N_n \otimes 1_\circ) \text{ is nilpotent}\}$$

An observation is said to be *positive* (resp. *boolean*) when for all i, j,

$$\phi_{i,j} = \sum_{l=0}^{m} \alpha_l \lambda(g_l) \text{ with } \alpha_l \geqslant 0 \text{ (resp. with } \alpha_l = 1)$$

We then define:

$$\begin{aligned} P_{\geqslant 0} &= \{\phi \mid \phi \text{ is a positive observation}\} \\ P_+ &= \{\phi \mid \phi \text{ is a boolean observation}\} \end{aligned}$$

$$\{P\} = \{[\phi] \mid \phi \in P\}$$

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•Positive or boolean observations

No interference between the “branches” of the computation.

$\hookrightarrow$ Non-deterministic computation.

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  All “branches” must reach 0 for the computation to stop.

  $\hookrightarrow$ Characterization of the complementary of a complexity class.

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•Crossed product with the group of finite permutations

Permutations to chose where the bit currently read is stored.

$\hookrightarrow$ The bit is stored only when the pointer moves.

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+ circular input

co-NL $\subseteq$ $\{ANDPM\} \subseteq \{NPPM\}$ $\subseteq$ $\{P_+\} \subseteq \{P_{\leqslant 0}\}$ $\subseteq$ **co-NL**

(A)NDPM Observations

Theorem

*A NDPM can decides a **co-NL** complete problem.*

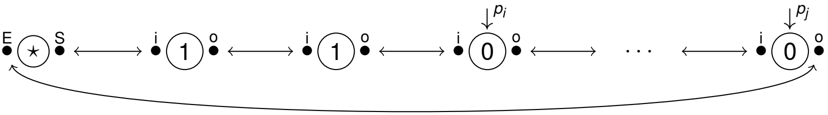
Definition (Non-Deterministic Pointer Machines)

A non-deterministic pointer machine (NDPM) with $p \in \mathbb{N}$ pointers is a triplet $M = \{Q, \Sigma, \to\}$ where

- Q is the set of *states*, $Q = \{\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_e\}$;
- $\Sigma = \{0, 1, \star\}$ is the *alphabet*;
- $\to \subseteq (\Sigma^p \times Q) \times (\wp((P^p \times Q) \backslash \emptyset) \cup \{\textbf{accept, reject}\})$ is the binary transition relation.

where P is the set of instructions: $P = \{p_i-, \epsilon_i, p_i+ \mid i \in \{1, \ldots, p\}\}$.

- Fixed (constant) number of pointers
- No access to the adresses
- Non-determinist



Theorem

There exists a NMDP that decides s-t-conn-Comp, a ***co-NL*** *complete problem.*

Definition

Let {NDPM} (resp. {ANDPM}) be the class of sets S such that there exists a NDPM (resp. an acyclic NDPM) that decides S.

Corollary

$$\textbf{\textit{co-NL}} \subseteq \{NDPM\}$$

$$\textbf{co-NL} \quad \subseteq \quad \{ANDPM\} = \{NPPM\} \quad \subseteq \quad \{P_+\} \subseteq \{P_{\leqslant 0}\} \quad \subseteq \quad \textbf{co-NL}$$

(A)NDPM Observations

Theorem

A ANDPM can be encoded in an observation.

$$(\mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

Definition

A *configuration* (resp. a *pseudo-configuration*) is an element of the set $n^p \times \Sigma^p \times Q$ (resp. $\Sigma^p \times Q$). The set of all possible pseudo-configurations of a NDPM M is denoted c_M.

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Definition (Acyclicity)

A NDPM M is said to be *acyclic* when for all $c \in C_M$ and all entry $n \in \mathbb{N}$, $M_c(n)$ halts.

Lemma

For all NDPM M that decides a set S there exists an acyclic NDPM M' that decides S.

Proposition (Encoding M_c)

- $\rightarrow^\bullet = \sum_{c \in C_M} \sum_{t \text{ s.t. } c \rightarrow t} \phi_{c,t}$
- *$Q^\bullet$ is in the matrix algebra.*
- *$P^\bullet$ by means of projections and permutations.*
- ***accept***$^\bullet = 0$
- ***reject***$^\bullet =$ *"restore M with initial pseudo-configuration c"*

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- ***accept***$^{\bullet} = 0$
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Theorem

For any acyclic NDPM M and pseudo-configuration $c \in C_M$*, there exists an observation* $M_c^{\bullet} \in \mathfrak{M}_6(\mathfrak{O}) \otimes \mathfrak{Q}_M$ *such that for all* $N_n \in \mathfrak{M}_6(\mathfrak{N}_0)$

$$M_c(n) \text{ accepts iff } M_c^{\bullet}(N_n \otimes 1_{\mathfrak{Q}_M}) \text{ is nilpotent.}$$

Moreover, $M_c^{\bullet} \in P_+$.

Proposition

$$\textbf{co-NL} \subseteq \{ANDPM\} = \{NDPM\} \subseteq \{P_+\} \subseteq \{P_{\geqslant 0}\}$$

$$\textbf{co-NL} \subseteq \{ANDPM\} = \{NPPM\} \subseteq \{P_+\} \subseteq \{P_{\leqslant 0}\} \subseteq \textbf{co-NL}$$

(A)NDPM — Observations

Theorem

A Turing Machine can decide if an observation accepts.

Lemma

There exist a morphism Φ and two matrices M and $\bar{\phi}$ such that $\Phi(M \otimes 1_{\mathfrak{E}}) = N_n \otimes 1_{\mathfrak{E}}$ and $\Phi(\bar{\phi}) = \phi$. So we have $\phi(N_n \otimes 1_{\mathfrak{E}})$ nilpotent if and only if $(M \otimes 1_{\mathfrak{E}})\bar{\phi}$ nilpotent.

Remark

It is equivalent to consider

$$\mathfrak{M}_6(\mathfrak{G}) \otimes \underbrace{\mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C})}_{p \text{ times}} \otimes \mathfrak{M}_k(\mathbb{C})$$

and

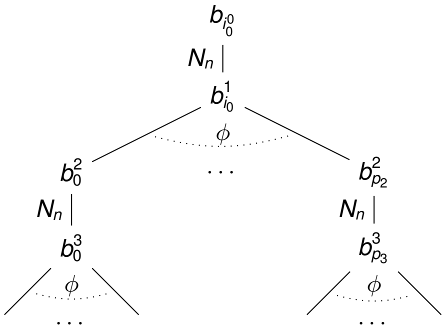
$$\mathfrak{M}_6(\mathbb{C}) \otimes ((\underbrace{\mathfrak{M}_{n+1}(\mathbb{C}) \otimes \cdots \otimes \mathfrak{M}_{n+1}(\mathbb{C})}_{p \text{ times}}) \rtimes G_p) \otimes \mathfrak{E}$$

whose basis contains elements of the form

$$(\pi, a_1, \ldots, a_p; \sigma; e)$$

Checking the nilpotency in **co-NL**

$$\bar{\phi}(\pi, a_1, \ldots, a_p; \sigma; e) = \sum_{i=0}^{K} \alpha_i(\rho, a_{\tau_i(1)}, \ldots, a_{\tau_i(p)}; \tau_i\sigma; e_i)$$



With b_i^j the elements of the basis encountered.

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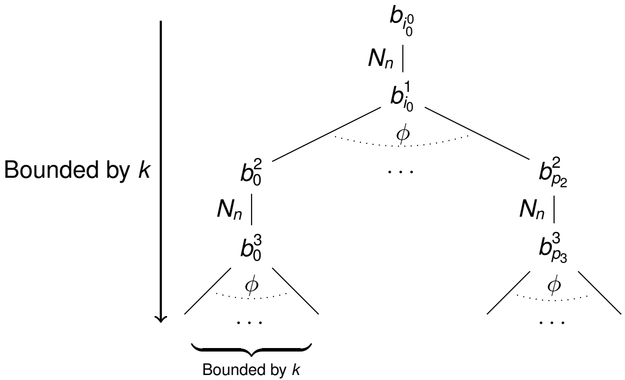

With b_i^j the elements of the basis encountered and k the dimensions of the underlying space, $6(n+1)^p p! d$ where d is the dimension of $\mathfrak{E}$.

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(A)NDPM   Observations

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