Characterizing **co-NL** by a Group Action

Focus Meeting

Clément Aubert
Joint work with Thomas Seiller (LAMA - Univ. de Savoie)

Institut Galilée - Université Paris-Nord
99, avenue Jean-Baptiste Clément
93430 Villetaneuse
`aubert@lipn.fr`

7 novembre 2012

$$\textbf{co-NL} \quad = \quad \{\mathit{ANDPM}\} = \{\mathit{NDPM}\} \quad = \quad \{P_+\} = \{P_{\geqslant 0}\} \quad = \quad \textbf{co-NL}$$

(A)NDPM

Observations

$$\textbf{co-NL} \quad \subseteq \quad \{ANDPM\} \subseteq \{NDPM\} \quad \subseteq \quad \{P_+\} \subseteq \{P_{\geqslant 0}\} \quad \subseteq \quad \textbf{co-NL}$$

(A)NDPM

Observations

$$(\mathfrak{M}_6(\mathfrak{S}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C})\,)\,(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

Our framework: an algebra

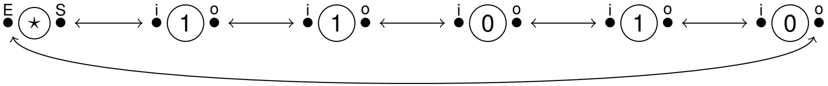
$$\underbrace{(\mathfrak{M}_6(\mathfrak{G}) \otimes \overbrace{\mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C})}^{=\mathfrak{O}})}_{\text{Observation}} \underbrace{(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})}_{\text{The input}}$$

Definition (Observations)

Let $(\mathfrak{N}_0, \mathfrak{G})$ be a normative pair. An *observation* is an operator $\phi \in \mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{O}$, where $\mathfrak{O}$ is a matrix algebra, i.e. $\mathfrak{O} = \mathfrak{M}_d(\mathbb{C})$ for $d \in \mathbb{N}$, called the *algebra of states*.

Definition (Binary representation of integers)

An operator $N_n \in \mathfrak{M}_6(\mathfrak{N}_0)$ is a *binary representation* of an integer n if ...



Our framework: an algebra

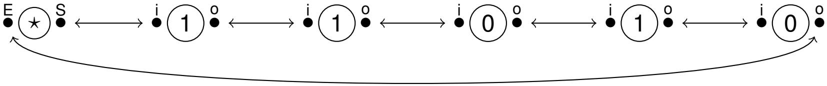
$$\underbrace{(\mathfrak{M}_6(\mathfrak{S}) \otimes \overbrace{\mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C})}^{=\mathfrak{O}})}_{\text{Observation}} \underbrace{(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})}_{\text{The input}}$$

Definition (Observations)

Let $(\mathfrak{N}_0, \mathfrak{S})$ be a normative pair. An *observation* is an operator $\phi \in \mathfrak{M}_6(\mathfrak{S}) \otimes \mathfrak{O}$, where $\mathfrak{O}$ is a matrix algebra, i.e. $\mathfrak{O} = \mathfrak{M}_d(\mathbb{C})$ for $d \in \mathbb{N}$, called the *algebra of states*.

Definition (Binary representation of integers)

An operator $N_n \in \mathfrak{M}_6(\mathfrak{N}_0)$ is a *binary representation* of an integer n if ...



The computation ends if $\exists k \in \mathbb{N}$ such that

$$(\phi(N_n \otimes 1_\circ))^k = 0$$

$$(\mathfrak{M}_6(\mathfrak{S}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

Definition (Normative Pairs)

Let $\mathfrak{N}_0$ and $\mathfrak{S}$ be two subalgebras of a von Neumann algebra $\mathfrak{M}$. The pair $(\mathfrak{N}_0, \mathfrak{S})$ is a *normative pair (in $\mathfrak{M}$)* if:

- $\mathfrak{N}_0$ is isomorphic to $\mathfrak{R}$;
- For all $\Phi \in \mathfrak{M}_6(\mathfrak{S})$ and $N_n, N'_n \in \mathfrak{M}_6(\mathfrak{N}_0)$ two binary representations of n,

$$\Phi N_n \text{ is nilpotent} \Leftrightarrow \Phi N'_n \text{ is nilpotent}$$

$$(\mathfrak{M}_6(\mathfrak{S}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

Definition (Normative Pairs)

Let $\mathfrak{N}_0$ and $\mathfrak{G}$ be two subalgebras of a von Neumann algebra $\mathfrak{M}$. The pair $(\mathfrak{N}_0, \mathfrak{G})$ is a *normative pair (in $\mathfrak{M}$)* if:

- $\mathfrak{N}_0$ is isomorphic to $\mathfrak{R}$;
- For all $\Phi \in \mathfrak{M}_6(\mathfrak{G})$ and $N_n, N'_n \in \mathfrak{M}_6(\mathfrak{N}_0)$ two binary representations of n,

$$\Phi N_n \text{ is nilpotent} \Leftrightarrow \Phi N'_n \text{ is nilpotent}$$

Proposition

Let G be the group of finite permutations over $\mathbb{N}$, α an action of G and for all $n \in \mathbb{N}$, $\mathfrak{N}_n = \mathfrak{R}$. The algebra$(\otimes_{n\in\mathbb{N}}\mathfrak{N}_s) \rtimes_{\hat{\alpha}} G$ contains a subalgebra generated by G that we will denote $\mathfrak{G}$.
$(\mathfrak{N}_0, \mathfrak{G})$ *is a normative pair in* $(\otimes_{n\in\mathbb{N}}\mathfrak{N}_s) \rtimes_{\hat{\alpha}} G$ *(the type II_1 hyperfinite factor).*

$$(\mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

Definition ($P_{\geqslant 0}$ and P_+)

Let $(\mathfrak{N}_0, \mathfrak{G})$ be a normative pair, $(\phi_{i,j})_{0 \leqslant i,j \leqslant 6d} \in \mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_d(\mathbb{C})$ an observation , we define:

$$[\phi] = \{n \in \mathbb{N} \mid \phi(N_n \otimes 1_\circ) \text{ is nilpotent}\}$$

An observation is said to be *positive* (resp. *boolean*) when for all i, j,

$$\phi_{i,j} = \sum_{l=0}^{m} \alpha_l \lambda(g_l) \text{ with } \alpha_l \geqslant 0 \text{ (resp. with } \alpha_l = 1)$$

We then define:

$$\begin{aligned} P_{\geqslant 0} &= \{\phi \mid \phi \text{ is a positive observation}\} \\ P_+ &= \{\phi \mid \phi \text{ is a boolean observation}\} \end{aligned}$$

$$\{P\} = \{[\phi] \mid \phi \in P\}$$

$$(\mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

•Positive or boolean observations

No interference between the “branches” of the computation.

$\hookrightarrow$ Non-deterministic computation.

$$(\mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

•Positive or boolean observations

No interference between the “branches” of the computation.

$\hookrightarrow$ Non-deterministic computation.

•Nilpotency

All “branches” must reach 0 for the computation to stop.

$\hookrightarrow$ Characterization of the complementary of a complexity class.

$$(\mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

•Positive or boolean observations

No interference between the “branches” of the computation.

$\hookrightarrow$ Non-deterministic computation.

•Nilpotency

All “branches” must reach 0 for the computation to stop.

$\hookrightarrow$ Characterization of the complementary of a complexity class.

•Crossed product with the group of finite permutations

Permutations to chose where the bit currently read is stored.

$\hookrightarrow$ The bit is stored only when the pointer moves.

$$(\mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

•Positive or boolean observations

No interference between the “branches” of the computation.

↪ Non-deterministic computation.

•Nilpotency

All “branches” must reach 0 for the computation to stop.

↪ Characterization of the complementary of a complexity class.

•Crossed product with the group of finite permutations

Permutations to chose where the bit currently read is stored.

↪ The bit is stored only when the pointer moves.

+ circular input

co-NL $\subseteq$ $\{ANDPM\} \subseteq \{NDPM\}$ $\subseteq$ $\{P_+\} \subseteq \{P_{\geqslant 0}\}$ $\subseteq$ **co-NL**

(A)NDPM Observations

Theorem

*A NDPM can decides a **co-NL** complete problem.*

co-NL $\subseteq$ $\{ANDPM\} \subseteq \{NDPM\}$ $\subseteq$ $\{P_+\} \subseteq \{P_{\geqslant 0}\}$ $\subseteq$ **co-NL**

(A)NDPM Observations

Theorem

A NDPM can decides a ***co-NL*** *complete problem.*

Remark (Arnaud Durand)

In fact, NDPM are (slightly modified) 2NFA(k), and it is proven that **NL** $= \cup_{k \geqslant 1} \mathcal{L}(2NFA(k))$.

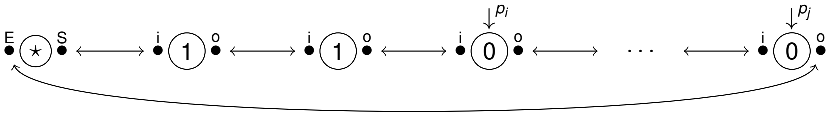
Definition (Non-Deterministic Pointer Machines)

A non-deterministic pointer machine (NDPM) with $p \in \mathbb{N}$ pointers is a triplet $M = \{Q, \Sigma, \rightarrow\}$ where

- Q is the set of *states*, $Q = \{\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_e\}$;
- $\Sigma = \{0, 1, \star\}$ is the *alphabet*;
- $\rightarrow \subseteq (\Sigma^p \times Q) \times (\wp((P^p \times Q) \setminus \emptyset) \cup \{\textbf{accept, reject}\})$ is the binary transition relation.

where P is the set of instructions: $P = \{p_i-, \epsilon_i, p_i+ \mid i \in \{1, \ldots, p\}\}$.

- Fixed (constant) number of pointers
- No access to the adresses
- Non-determinist



Theorem

There exists a NMDP that decides s-t-conn-Comp, a ***co-NL*** *complete problem.*

Definition

Let $\{\text{NDPM}\}$ (resp. $\{\text{ANDPM}\}$) be the class of sets S such that there exists a NDPM (resp. an acyclic NDPM) that decides S.

Corollary

$$\textbf{\textit{co-NL}} \subseteq \{NDPM\}$$

co-NL $\subseteq$ $\{ANDPM\} = \{NDPM\}$ $\subseteq$ $\{P_+\} \subseteq \{P_{\geqslant 0}\}$ $\subseteq$ **co-NL**

(A)NDPM   Observations

Theorem

A ANDPM can be encoded in an observation.

$$(\mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

Definition

A *configuration* (resp. a *pseudo-configuration*) is an element of the set $n^p \times \Sigma^p \times Q$ (resp. $\Sigma^p \times Q$). The set of all possible pseudo-configurations of a NDPM M is denoted c_M.

$$(\mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

Definition

A *configuration* (resp. a *pseudo-configuration*) is an element of the set $n^p \times \Sigma^p \times Q$ (resp. $\Sigma^p \times Q$). The set of all possible pseudo-configurations of a NDPM M is denoted c_M.

Definition (Acyclicity)

A NDPM M is said to be *acyclic* when for all $c \in C_M$ and all entry $n \in \mathbb{N}$, $M_c(n)$ halts.

Lemma

For all NDPM M that decides a set S there exists an acyclic NDPM M' that decides S.

$$(\mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

Proposition (Encoding M_c)

- $\rightarrow^\bullet = \sum_{c \in C_M} \sum_{t \text{ s.t. } c \rightarrow t} \phi_{c,t}$
- $Q^\bullet$ *is in the matrix algebra.*
- $P^\bullet$ *by means of projections and permutations.*
- ***accept***$^\bullet = 0$
- ***reject***$^\bullet =$ *"restore M with initial pseudo-configuration c"*

$$(\mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C}))(\mathfrak{M}_6(\mathfrak{N}_0) \otimes \mathfrak{O})$$

Proposition (Encoding M_c)

- $\rightarrow^\bullet = \sum_{c \in C_M} \sum_{t \text{ s.t. } c \rightarrow t} \phi_{c,t}$
- $Q^\bullet$ *is in the matrix algebra.*
- $P^\bullet$ *by means of projections and permutations.*
- ***accept***$^\bullet = 0$
- ***reject***$^\bullet =$ *"restore M with initial pseudo-configuration c"*

Theorem

For any acyclic NDPM M and pseudo-configuration $c \in C_M$, there exists an observation $M_c^\bullet \in \mathfrak{M}_6(\mathfrak{O}) \otimes \mathfrak{Q}_M$ such that for all $N_n \in \mathfrak{M}_6(\mathfrak{N}_0)$

$$M_c(n) \text{ accepts iff } M_c^\bullet(N_n \otimes 1_{\mathfrak{Q}_M}) \text{ is nilpotent.}$$

Moreover, $M_c^\bullet \in P_+$.

$$\textbf{co-NL} \quad \subseteq \quad \{ANDPM\} = \{NDPM\} \quad \subseteq \quad \{P_{+}\} \subseteq \{P_{\geqslant 0}\} \quad \subseteq \quad \textbf{co-NL}$$

(A)NDPM Observations

Theorem

A Turing Machine can decide if an observation accepts.

Lemma

There exist a morphism Φ *and two matrices* M *and* $\bar{\phi}$ *such that* $\Phi(M \otimes 1_{\mathfrak{E}}) = N_n \otimes 1_{\mathfrak{E}}$ *and* $\Phi(\bar{\phi}) = \phi$*. So we have* $\phi(N_n \otimes 1_{\mathfrak{E}})$ *nilpotent if and only if* $(M \otimes 1_{\mathfrak{E}})\bar{\phi}$ *nilpotent.*

Remark

It is equivalent to consider

$$\mathfrak{M}_6(\mathfrak{G}) \otimes \underbrace{\mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C})}_{p \text{ times}} \otimes \mathfrak{M}_k(\mathbb{C})$$

and

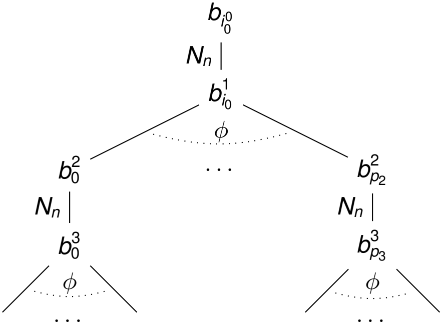
$$\mathfrak{M}_6(\mathbb{C}) \otimes ((\underbrace{\mathfrak{M}_{n+1}(\mathbb{C}) \otimes \cdots \otimes \mathfrak{M}_{n+1}(\mathbb{C})}_{p \text{ times}}) \rtimes G_p) \otimes \mathfrak{E}$$

whose basis contains elements of the form

$$(\pi, a_1, \ldots, a_p; \sigma; e)$$

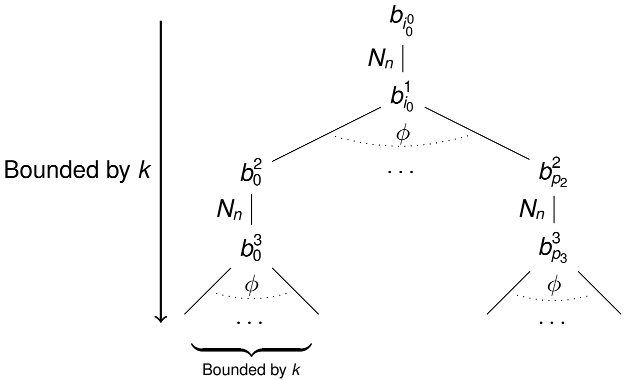
Checking the nilpotency in **co-NL**

$$\bar{\phi}(\pi, a_1, \ldots, a_p; \sigma; e) = \sum_{i=0}^{K} \alpha_i(\rho, a_{\tau_i(1)}, \ldots, a_{\tau_i(p)}; \tau_i\sigma; e_i)$$



With b_i^j the elements of the basis encountered.

$$\bar{\phi}(\pi, a_1, \ldots, a_p; \sigma; e) = \sum_{i=0}^{K} \alpha_i(\rho, a_{\tau_i(1)}, \ldots, a_{\tau_i(p)}; \tau_i\sigma; e_i)$$



With b_i^j the elements of the basis encountered and k the dimensions of the underlying space, $6(n+1)^p p! d$ where d is the dimension of $\mathfrak{E}$.

$$\textbf{co-NL} \quad = \quad \{ANDPM\} = \{NDPM\} \quad = \quad \{P_+\} = \{P_{\geqslant 0}\} \quad = \quad \textbf{co-NL}$$

(A)NDPM Observations

lipn.fr/~aubert/
aubert@lipn.fr