Characterizing **co-NL** by a Group Action

Séminaire Logique et Interactions

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$$\textbf{co-NL} \quad = \quad \{\mathit{ANDPM}\} = \{\mathit{NDPM}\} \quad = \quad \{P_+\} = \{P_{\geqslant 0}\} \quad = \quad \textbf{co-NL}$$

(A)NDPM

Observations

$$\textbf{co-NL} \quad \subseteq \quad \{ANDPM\} \subseteq \{NDPM\} \quad \subseteq \quad \{P_+\} \subseteq \{P_{\geqslant 0}\} \quad \subseteq \quad \textbf{co-NL}$$

(A)NDPM

Observations

Definition (Normative Pairs)

Let $\mathfrak{N}_0$ and $\mathfrak{S}$ be two subalgebras of a von Neumann algebra $\mathfrak{M}$. The pair $(\mathfrak{N}_0, \mathfrak{S})$ is a *normative pair (in $\mathfrak{M}$)* if:

- $\mathfrak{N}_0$ is isomorphic to $\mathfrak{R}$;
- For all $\Phi \in \mathfrak{M}_6(\mathfrak{S})$ and $N_n, N_n' \in \mathfrak{M}_6(\mathfrak{N}_0)$ two binary representations of n,

$$\Phi N_n \text{ is nilpotent} \Leftrightarrow \Phi N_n' \text{ is nilpotent}$$

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Proposition

Let G be the group of finite permutations over $\mathbb{N}$, α an action of G and for all $n \in \mathbb{N}$, $\mathfrak{N}_n = \mathfrak{R}$. The algebra$(\otimes_{n\in\mathbb{N}}\mathfrak{N}_s) \rtimes_{\hat{\alpha}} G$ contains a subalgebra generated by G that we will denote $\mathfrak{G}$.
$(\mathfrak{N}_0, \mathfrak{G})$ is a normative pair in $(\otimes_{n\in\mathbb{N}}\mathfrak{N}_s) \rtimes_{\hat{\alpha}} G$ (the type II_1 hyperfinite factor).

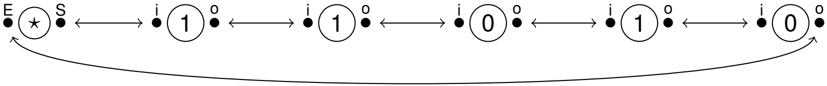
Our framework: an algebra

Definition (Observations)

Let $(\mathfrak{N}_0, \mathfrak{S})$ be a normative pair. An *observation* is an operator $\phi \in \mathfrak{M}_6(\mathfrak{S}) \otimes \mathfrak{Q}$, where $\mathfrak{Q}$ is a matrix algebra, i.e. $\mathfrak{Q} = \mathfrak{M}_d(\mathbb{C})$ for $d \in \mathbb{N}$, called the *algebra of states*.

Definition (Binary representation of integers)

An operator $N_n \in \mathfrak{M}_6(\mathfrak{N}_0)$ is a *binary representation* of an integer n if ...



The computation ends if $\exists k \in \mathbb{N}$ such that

$$(\phi(N_n \otimes 1_{\mathfrak{Q}}))^k = 0$$

Definition ($P_{\geqslant 0}$ and P_+)

We take $(\mathfrak{N}_0, \mathfrak{G})$ as normative pair and let $(\phi_{i,j})_{0 \leqslant i,j \leqslant 6d} \in \mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{Q}$ be an observation, we define:

$$[\phi] = \{n \in \mathbb{N} \mid \phi(N_n \otimes 1_{\mathfrak{Q}}) \text{ is nilpotent}\}$$

An observation is said to be *positive* (resp. *boolean*) when for all i, j,

$$\phi_{i,j} = \sum_{l=0}^{m} \alpha_l \lambda(g_l) \text{ with } \alpha_l \geqslant 0 \text{ (resp. with } \alpha_l = 1)$$

We then define:

$$\begin{aligned} P_{\geqslant 0} &= \{\phi \mid \phi \text{ is a positive observation}\} \\ P_+ &= \{\phi \mid \phi \text{ is a boolean observation}\} \end{aligned}$$

$$\{P\} = \{[\phi] \mid \phi \in P\}$$

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No interference between the “branches” of the computation.

$\hookrightarrow$ Non-deterministic computation.

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- Crossed product with the group of finite permutations
  Permutations to chose where the bit currently read is stored.
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Singularities of this framework

•Positive or boolean observations
No interference between the "branches" of the computation.
↪ Non-deterministic computation.

•Nilpotency
All "branches" must reach 0 for the computation to stop.
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+ circular input

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(A)NDPM Observations

Theorem

A NDPM can decide a ***co-NL*** *complete problem.*

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Remark (Arnaud Durand)

*In fact, NDPM are (slightly modified) 2NFA(k), and it is proven that **co-NL** $=$ **NL** $= \cup_{k\geqslant 1}\mathcal{L}(2NFA(k))$.*

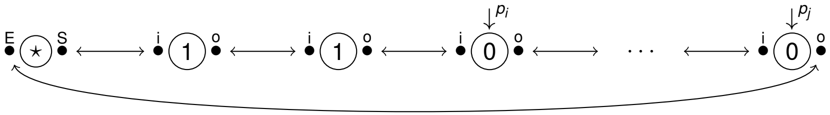
Definition (Non-Deterministic Pointer Machines)

A non-deterministic pointer machine (NDPM) with $p \in \mathbb{N}$ pointers is a triplet $M = \{Q, \Sigma, \to\}$ where

- Q is the set of *states*, $Q = \{\mathbf{q}_0, \mathbf{q}_1, \dots, \mathbf{q}_e\}$;
- $\Sigma = \{0, 1, \star\}$ is the *alphabet*;
- $\to \subseteq (\Sigma^p \times Q) \times (\wp((P^p \times Q) \setminus \emptyset) \cup \{\textbf{accept, reject}\})$ is the binary transition relation.

where P is the set of instructions: $P = \{p_i-, \epsilon_i, p_i+ \mid i \in \{1, \dots, p\}\}$.

- Fixed (constant) number of pointers
- No access to the adresses
- Non-deterministic



Theorem

There exists a NMDP that decides s-t-conn-Comp, a ***co-NL*** *complete problem.*

Definition

Let {NDPM} (resp. {ANDPM}) be the class of sets S such that there exists a NDPM (resp. an acyclic NDPM) that decides S.

Corollary

$$\textbf{\textit{co-NL}} \subseteq \{\textit{NDPM}\}$$

co-NL $\subseteq$ $\{ANDPM\} = \{NDPM\}$ $\subseteq$ $\{P_+\} \subseteq \{P_{\geqslant 0}\}$ $\subseteq$ **co-NL**

(A)NDPM — Observations

Theorem

A ANDPM can be encoded in an observation.

$$\mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{Q}_M = \mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C})$$

Definition

A *configuration* (resp. a *pseudo-configuration*) is an element of the set $n^p \times \Sigma^p \times Q$ (resp. $\Sigma^p \times Q$). The set of all possible pseudo-configurations of a NDPM M is denoted c_M.

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Definition (Acyclicity)

A NDPM M is said to be *acyclic* when for all $c \in C_M$ and all entry $n \in \mathbb{N}$, $M_c(n)$ halts.

Lemma

For all NDPM M that decides a set S there exists an acyclic NDPM M' that decides S.

$$\mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{Q}_M = \mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \ldots \otimes \mathfrak{M}_6(\mathbb{C}) \otimes \mathfrak{M}_k(\mathbb{C})$$

Proposition (Encoding M_c)

- $\rightarrow^{\bullet} = \sum_{c \in C_M} \sum_{t \text{ s.t. } c \rightarrow t} \phi_{c,t}$
- $Q^{\bullet}$ *is in the matrix algebra.*
- $P^{\bullet}$ *by means of projections and permutations.*
- $\textbf{\textit{accept}}^{\bullet} = 0$
- $\textbf{\textit{reject}}^{\bullet} =$ *“restore M with initial pseudo-configuration c”*

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- ***accept***$^{\bullet} = 0$
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Theorem

For any acyclic NDPM M and pseudo-configuration $c \in C_M$, there exists an observation $M_c^{\bullet} \in \mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{Q}_M$ such that for all $N_n \in \mathfrak{M}_6(\mathfrak{N}_0)$

$$M_c(n) \text{ accepts iff } M_c^{\bullet}(N_n \otimes 1_{\mathfrak{Q}_M}) \text{ is nilpotent.}$$

Moreover, $M_c^{\bullet} \in P_{+}$.

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(A)NDPM Observations

Theorem

A Turing Machine can decide if an observation accepts an integer.

Lemma

There exists an injective morphism ψ and two matrices M and $\bar{\phi}$ such that $\psi(M \otimes 1_{\mathfrak{E}}) = N_n \otimes 1_{\mathfrak{E}}$ and $\psi(\bar{\phi}) = \phi$. So we have $\phi(N_n \otimes 1_{\mathfrak{E}})$ nilpotent if and only if $(M \otimes 1_{\mathfrak{E}})\bar{\phi}$ nilpotent.

Remark

It is equivalent to consider

$$\mathfrak{M}_6(\mathbb{C}) \otimes (\otimes_{\infty}^{i=0} \mathfrak{N}_i \rtimes G_p) \otimes \mathfrak{E}$$

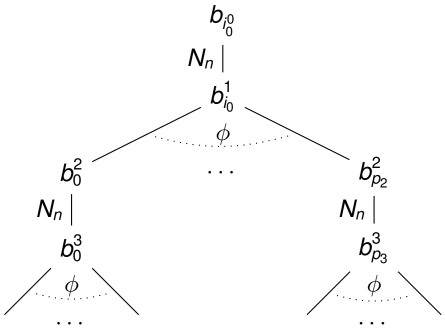
and

$$\mathfrak{M}_6(\mathbb{C}) \otimes (\otimes_{p}^{i=0} \mathfrak{M}_{n+1}(\mathbb{C}) \rtimes G_p \upharpoonright_{\{1,\dots,p\}}) \otimes \mathfrak{E}$$

which acts on a finite Hilbert space. We choose a basis for this space whose elements are of the form:

$$(\pi, a_1, \dots, a_p; \sigma; e)$$

$$\bar{\phi}(\pi, a_1, \ldots, a_p; \sigma; e) = \sum_{i=0}^{K} \alpha_i(\rho, a_{\tau_i(1)}, \ldots, a_{\tau_i(p)}; \tau_i\sigma; e_i)$$



With b_i^j the elements of the basis encountered.

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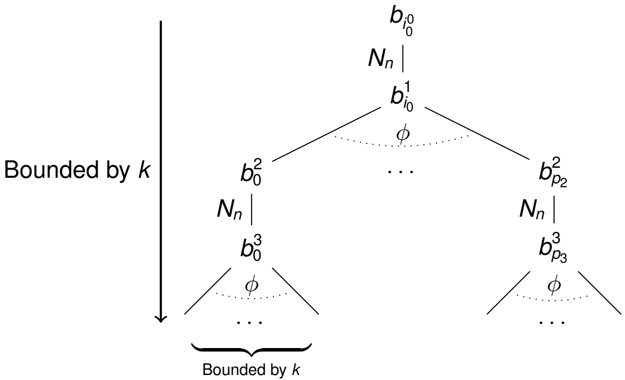

With b_i^j the elements of the basis encountered and k the dimensions of the underlying space, $6(n+1)^p p! d$ where d is the dimension of $\mathfrak{E}$.

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(A)NDPM Observations

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