Logarithmic Space and Permutations

LCC'13, Torino

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Joint work with Thomas Seiller

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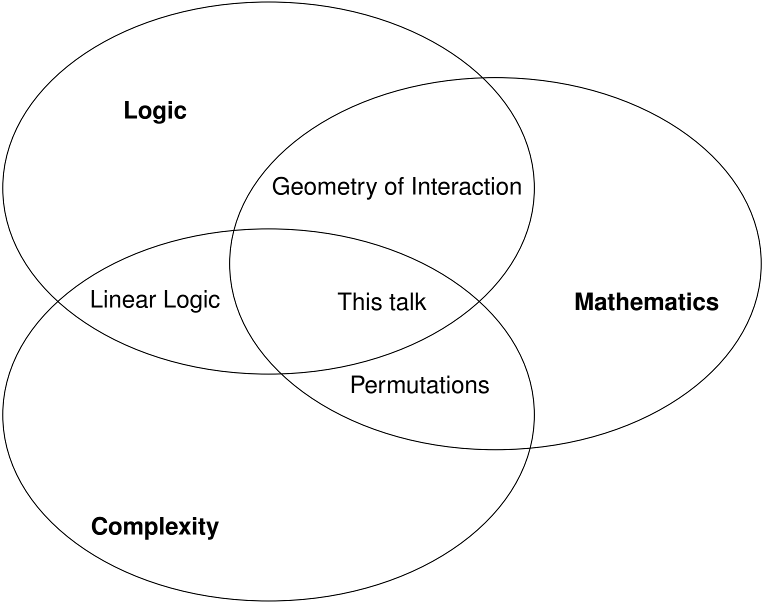
Clément Aubert and Thomas Seiller.
Characterizing co-NL by a group action.
CoRR, abs/1209.3422, 2012.

Clément Aubert and Thomas Seiller.
Logarithmic space and permutations.
CoRR, abs/1301.3189, 2013.

Jean-Yves Girard.
Normativity in logic.
In Peter Dybjer, Sten Lindström, Erik Palmgren, and Göran Sundholm, editors, *Epistemology versus Ontology*, volume 27 of *Logic, Epistemology, and the Unity of Science*, pages 243–263. Springer, 2012.


Logic
Geometry of Interaction
Linear Logic
This talk
Mathematics
Permutations
Complexity

Integers $\to$ Binary List $\to$ λ-term $\to$ proof $\to$ Proof-Net $\to$ Matrices

$0, 1, 2, 3, \dots$

$\hookrightarrow 001, 010, 011, 100, \dots$

$\hookrightarrow \lambda f_0 \lambda f_1 \lambda_x \cdot f_0(f_1(f_1(\dots(f_0 x)\dots)$

$\hookrightarrow \vdash \forall X(X \to X) \to ((X \to X) \to (X \to X))$

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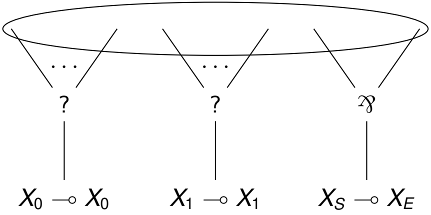

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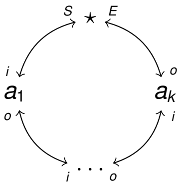
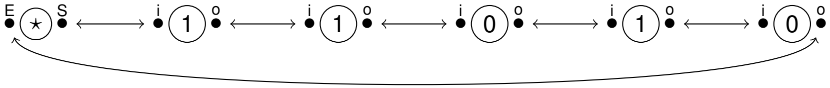


Integers $\rightarrow$ Binary List $\rightarrow$ λ-term $\rightarrow$ proof in L.L. $\rightarrow$ Proof-Net $\rightarrow$ Matrices

$$M_n = \begin{array}{c} \overbrace{\qquad\qquad}^{0} \; \overbrace{\qquad\qquad}^{1} \; \overbrace{\qquad\qquad}^{\star} \\ \begin{pmatrix} 0 & l_{00} & 0 & l_{10} & l_{S0} & 0 \\ {l_{00}}^t & 0 & {l_{01}}^t & 0 & 0 & {l_{0E}}^t \\ 0 & l_{01} & 0 & l_{11} & l_{S1} & 0 \\ {l_{10}}^t & 0 & {l_{11}}^t & 0 & 0 & {l_{1E}}^t \\ {l_{S0}}^t & 0 & {l_{S1}}^t & 0 & 0 & 0 \\ 0 & l_{0E} & 0 & l_{1E} & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \left.\right\} 0 \\ \left.\right\} 1 \\ \left.\right\} \star \end{array}$$

Definition (Binary representation of integers)

An operator $N_n \in \mathfrak{M}_6(\mathfrak{N}_0)$ is a *binary representation* of an integer n if ...



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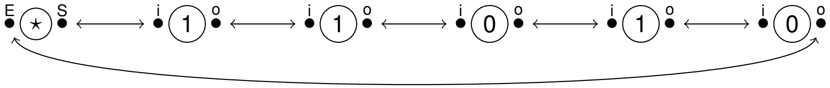

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An *observation* is an operator $\phi \in \mathfrak{M}_6(\mathfrak{S})$.

How to compute with integers as operators?

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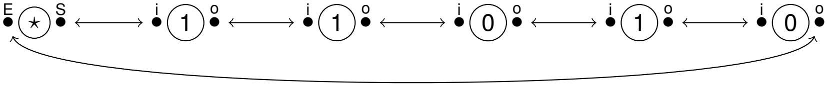

Definition (Observations)

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Definition (Computing, accepting)

The computation ends if $\exists k \in \mathbb{N}$ such that

$$(\phi(N_n))^k = 0$$

Definition (Normative Pairs)

Let $\mathfrak{N}_0$ and $\mathfrak{S}$ be two subalgebras of a von Neumann algebra $\mathfrak{M}$. The pair $(\mathfrak{N}_0, \mathfrak{S})$ is a *normative pair (in $\mathfrak{M}$)* if:

- $\mathfrak{N}_0$ is isomorphic to $\mathfrak{R}$;
- For all $\Phi \in \mathfrak{M}_6(\mathfrak{S})$ and $N_n, N_n' \in \mathfrak{M}_6(\mathfrak{N}_0)$ two binary representations of n,

$$\Phi N_n \text{ is nilpotent} \Leftrightarrow \Phi N_n' \text{ is nilpotent}$$

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Proposition

Let G be the group of finite permutations over $\mathbb{N}$, α an action of G and for all $n \in \mathbb{N}$, $\mathfrak{N}_n = \mathfrak{R}$. The algebra $(\otimes_{n\in\mathbb{N}}\mathfrak{N}_n) \rtimes_\alpha G$ contains a subalgebra generated by G that we will denote $\mathfrak{G}$.
$(\mathfrak{N}_0, \mathfrak{G})$ is a normative pair in $(\otimes_{n\in\mathbb{N}}\mathfrak{N}_n) \rtimes_\alpha G$.

Definition (P_+ and $P_{+,1}$)

Let $(\mathfrak{N}_0, \mathfrak{G})$ be a normative pair, $\phi \in \mathfrak{M}_6(\mathfrak{G})$ an observation, we define:

$$[\phi] = \{n \in \mathbb{N} \mid \phi(N_n) \text{ is nilpotent}\}$$

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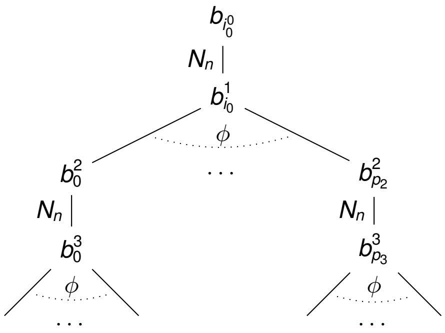
We will prove

$$\{P_+\} = \textbf{co-NL}$$
$$\{P_{+,1}\} = \textbf{L}$$

Checking the nilpotency in **co-NL**

If $\phi \in P_+$

$$\phi(\pi, a_1, \ldots, a_p; \sigma; e) = \sum_{i=0}^{K} \alpha_i(\rho, a_{\tau_i(1)}, \ldots, a_{\tau_i(p)}; \tau_i\sigma; e_i)$$



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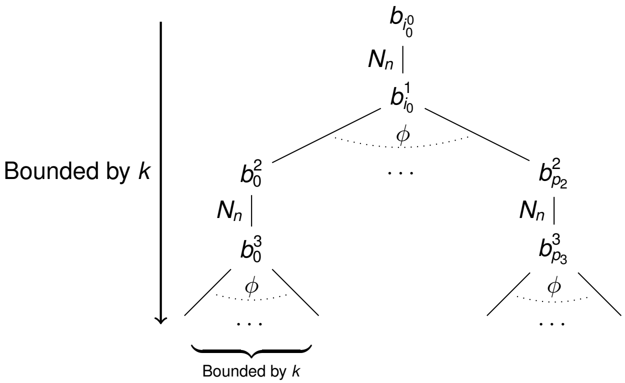

With b_i^j the elements of the basis encountered and k the dimensions of the underlying space.

Checking the nilpotency in **L**

If $\phi \in P_{+,1}$

$$\phi(\pi, a_1, \ldots, a_p; \sigma; e) = \qquad \alpha_i(\rho, a_{\tau_i(1)}, \ldots, a_{\tau_i(p)}; \tau_i\sigma; e_i)$$

Bounded by k $\downarrow$

$$\begin{array}{r} b_{i_0^0} \\ N_n \Big| \\ b_{i_0}^1 \\ \phi \Big| \\ b_0^2 \\ N_n \Big| \\ b_0^3 \\ \vdots \end{array}$$

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  Permutations to chose where the bit currently read is stored.
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+ **Technical details** : circular input, specific kind of initialization, etc.

But which one?

- Purple
- JAG
- Knuth's Linking Automaton
- Tarjan's Reference Machine
- SMM, KUM,
- 2NDFA

Theorem

$$\boldsymbol{NL} = \cup_{k \geqslant 1} \mathcal{L}(2NDFA(k))$$
$$\boldsymbol{L} = \cup_{k \geqslant 1} \mathcal{L}(2DFA(k))$$

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$$\textbf{\textit{co-NL}} = \cup_{k \geqslant 1} \mathcal{L}(NDPM(k))$$
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Theorem

For any NDPM M, there exists an observation $M^{\bullet} \in \mathfrak{M}_6(\mathfrak{G})$ such that for all $N_n \in \mathfrak{M}_6(\mathfrak{N}_0)$

$$M(n) \text{ accepts iff } M^{\bullet}(N_n) \text{ is nilpotent.}$$

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Theorem

$$\{P_{+}\} = \textbf{co-NL}$$
$$\{P_{+,1}\} = \textbf{L}$$

- Defined a representation of integers as operators
- Defined “observations” *i.e.* programs as operators
- Took a specific sub-algebra
- Checked that nilpotency could be decided with logarithmic resources
- Defined an encoding from 2NDFA to operators

- Finite matrices?
- Different constrain on norm, coefficient, etc.
- Another group?