Reversible computations are computations

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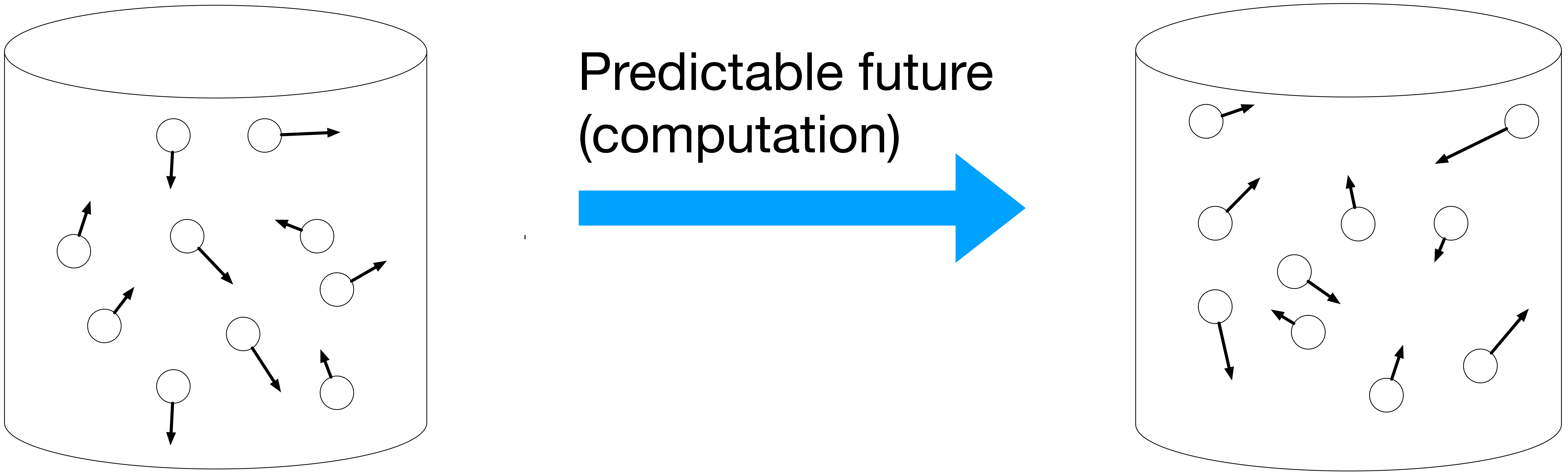
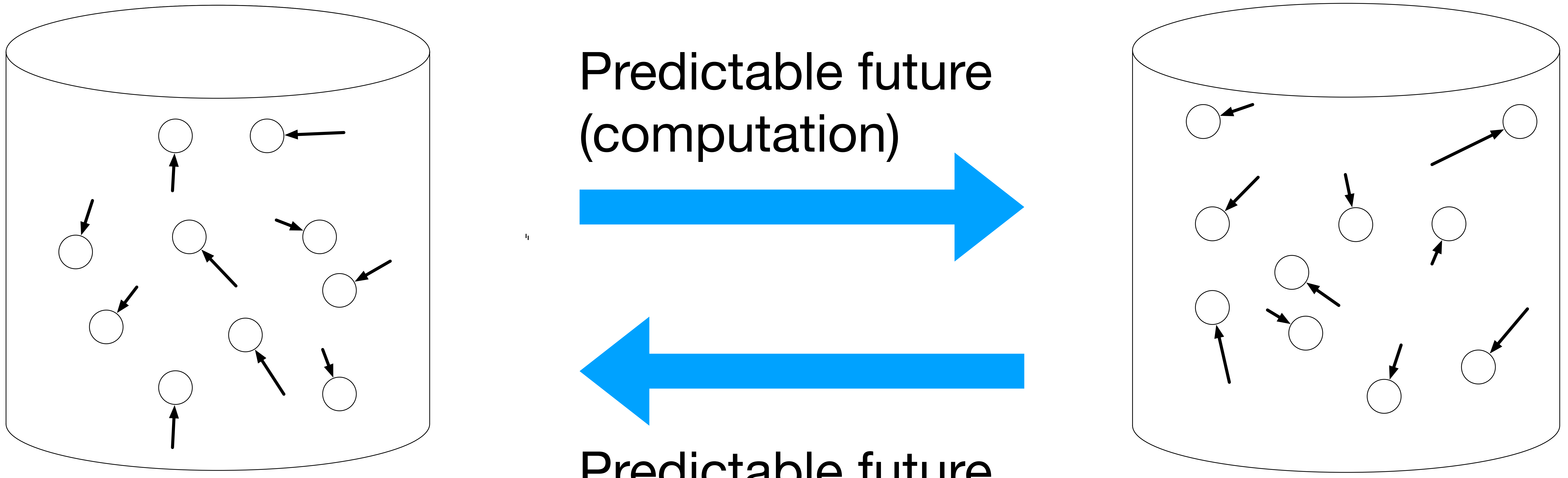
MFPS 2025

Dynamical Systems



Move, bounce,
local interactions

Dynamical Systems



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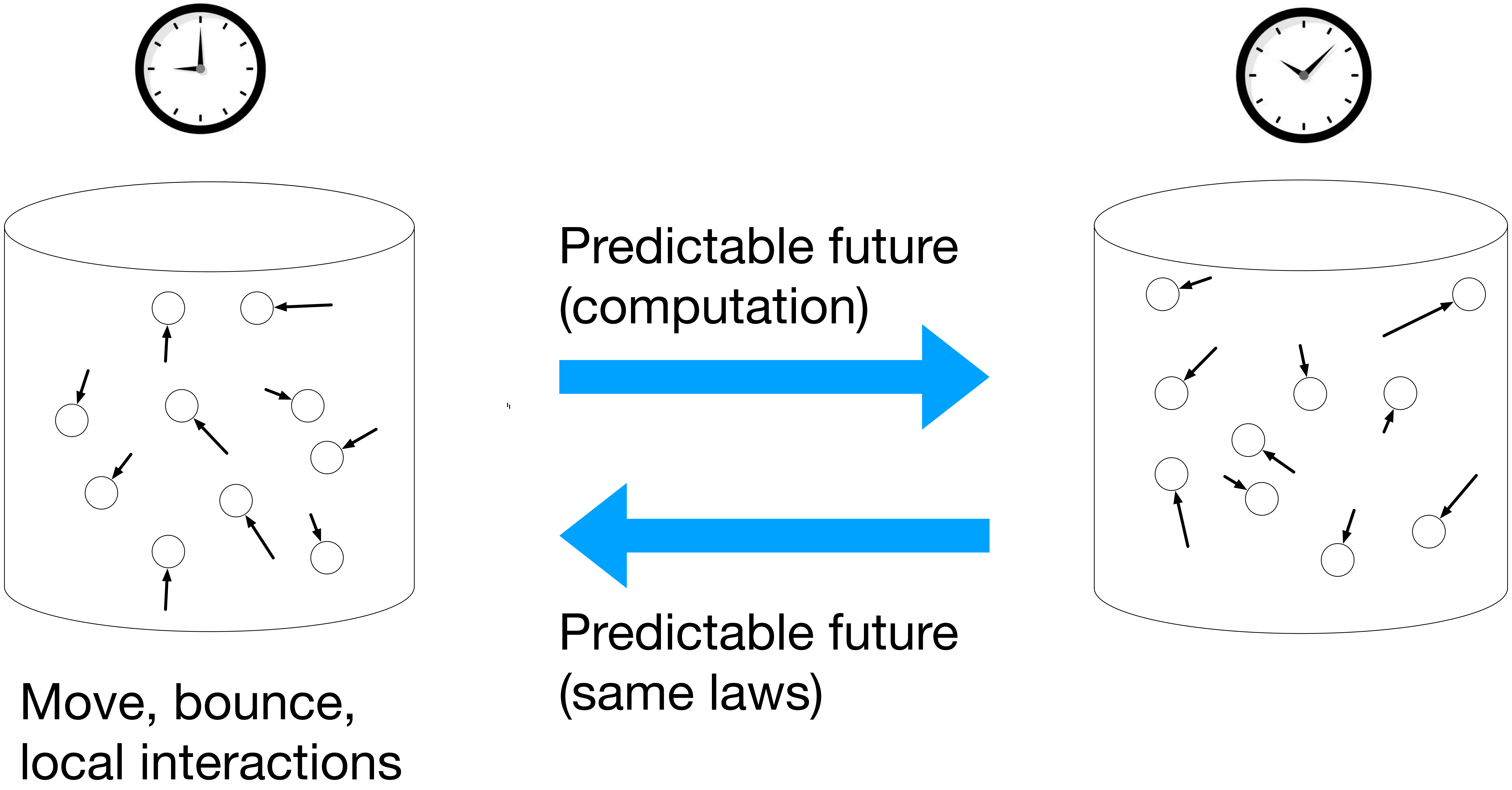
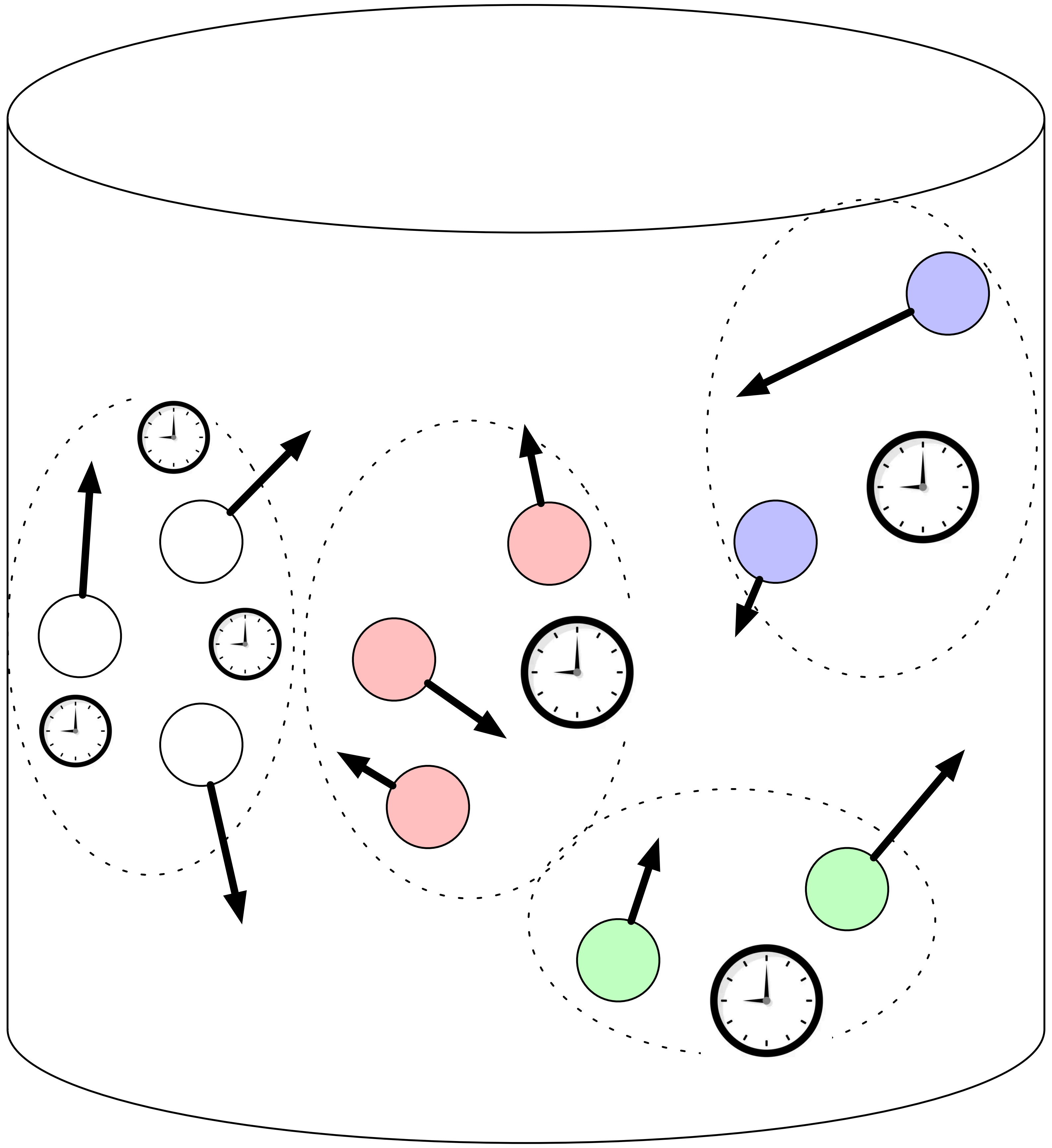
Dynamical Systems



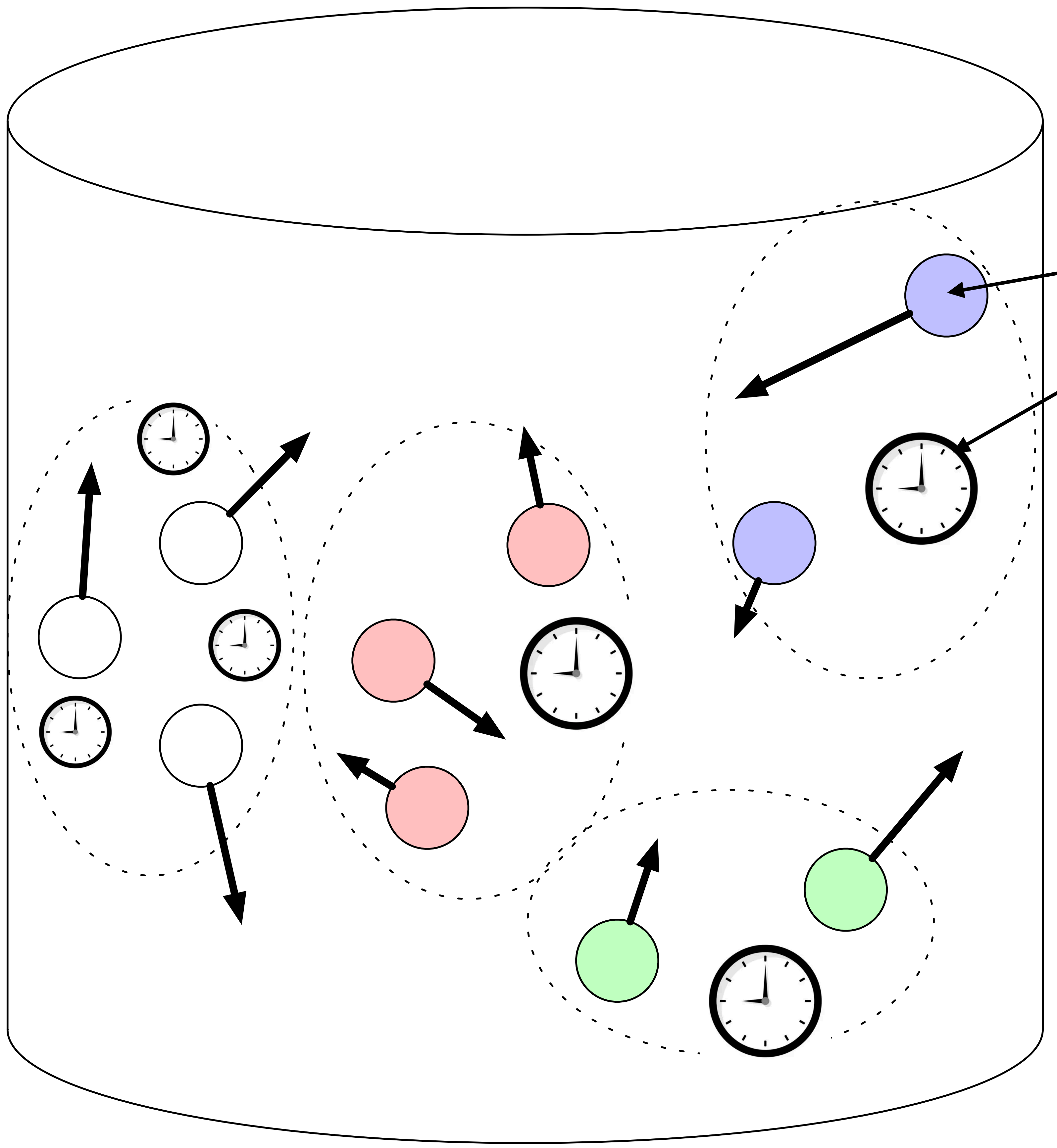
Causal consistent reversibility



A subsystem may be “backwards”, while the rest is “forwards”…

But agents can interact only if they are in compatible space-time

Causal consistent reversibility



Agent cannot backtrack beyond this time point alone to avoid a paradox…

A subsystem may be “backwards”, while the rest is “forwards”…

But agents can interact only if they are in compatible space-time

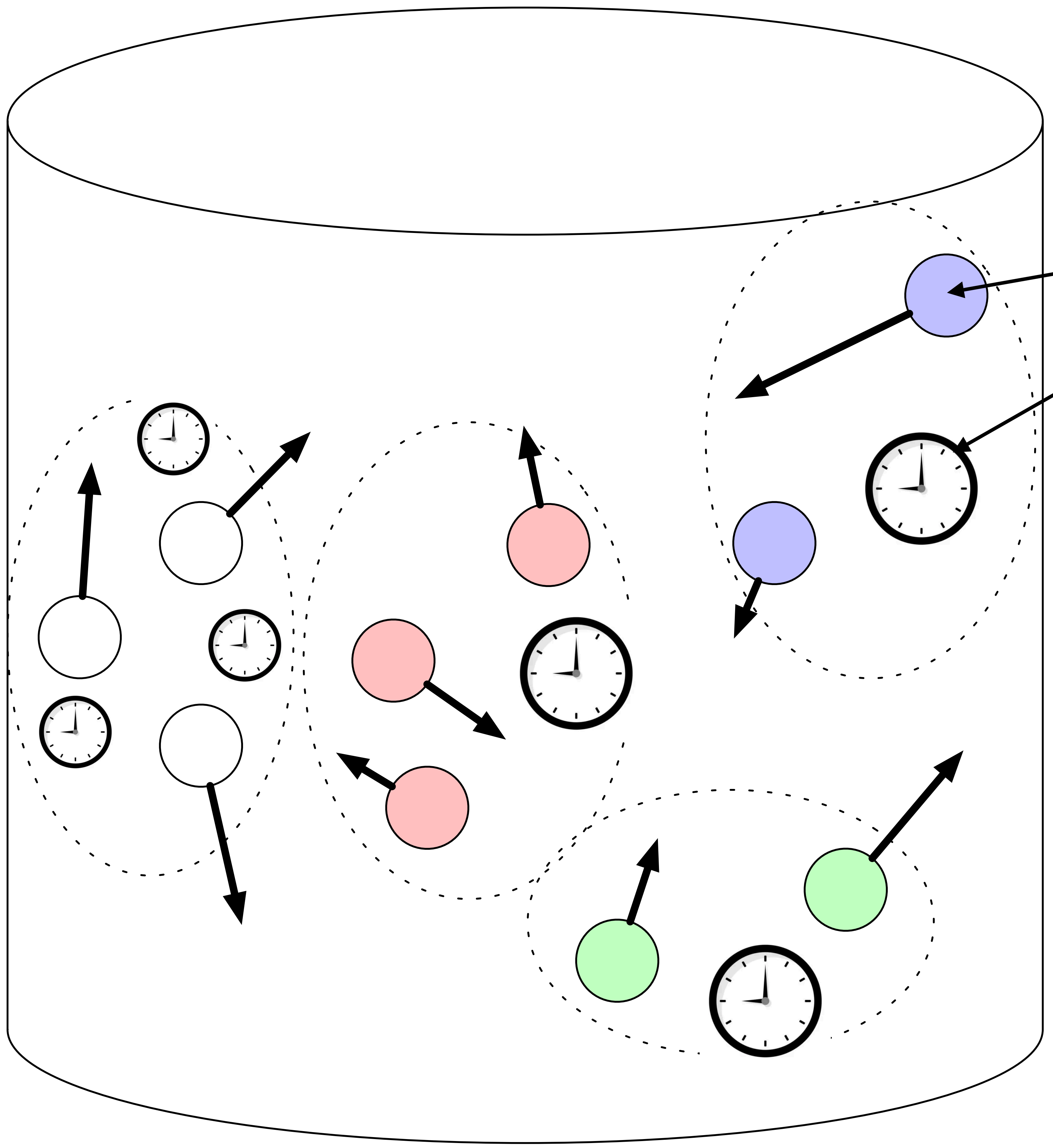
Causal consistent reversibility

Transactional systems

Concurrent debuggers

Counterfactual reasoning

Self assembly



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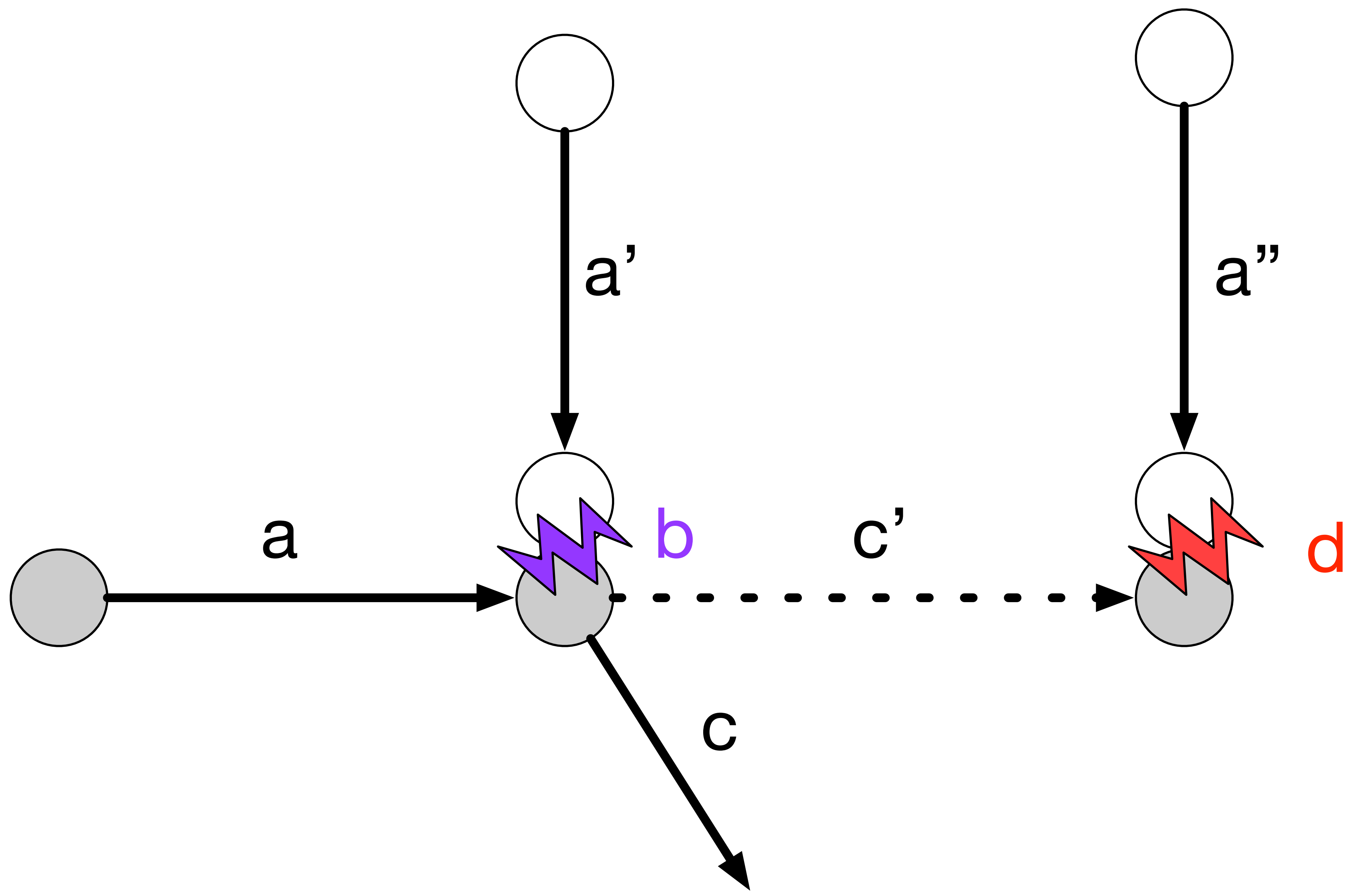
Overview



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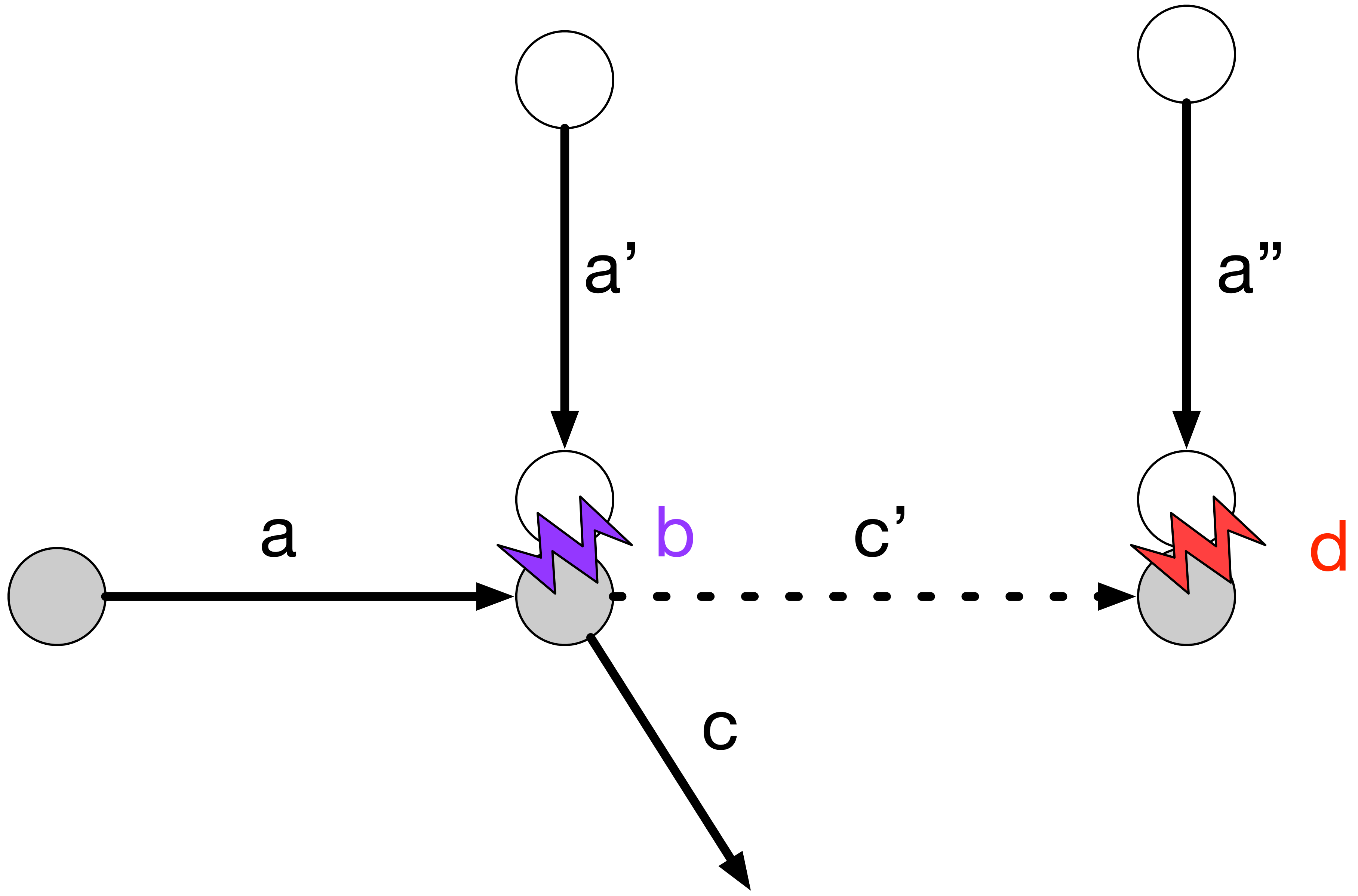
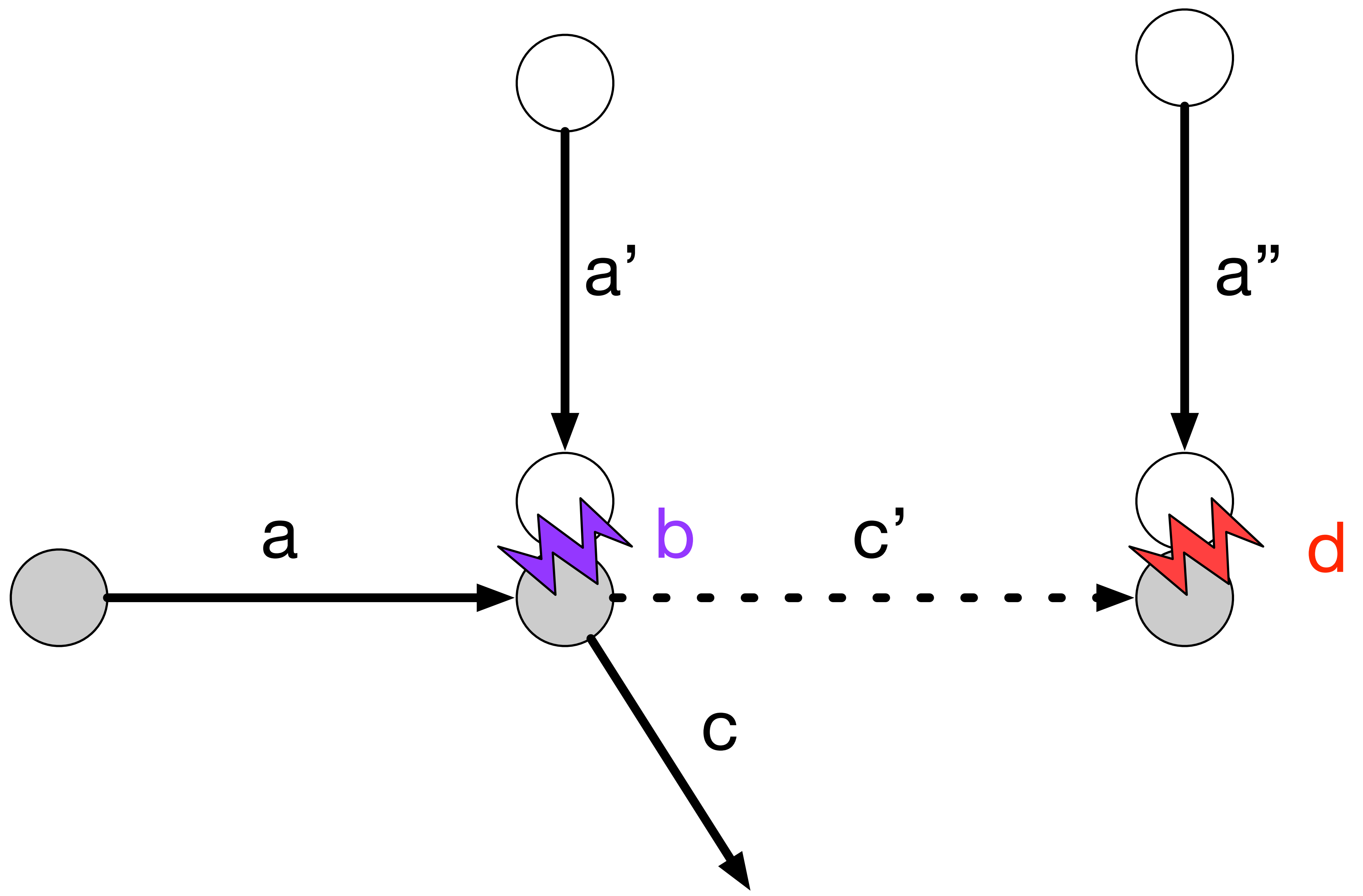


Logical characterisation

Time	$\{a, a'\} < b \;\; \wedge b < c$
$\varnothing$	$a' \# c' \quad a < c'$
	$\{a'', c'\} < d$

Overview



Logical characterisation

Time

$\varnothing$

$\{a, a'\} < b \;\; \wedge b < c$

$a' \# c' \quad a < c'$

$\{a'', c'\} < d$

Time

$\{a, c', a'', d\}$

$\varnothing$

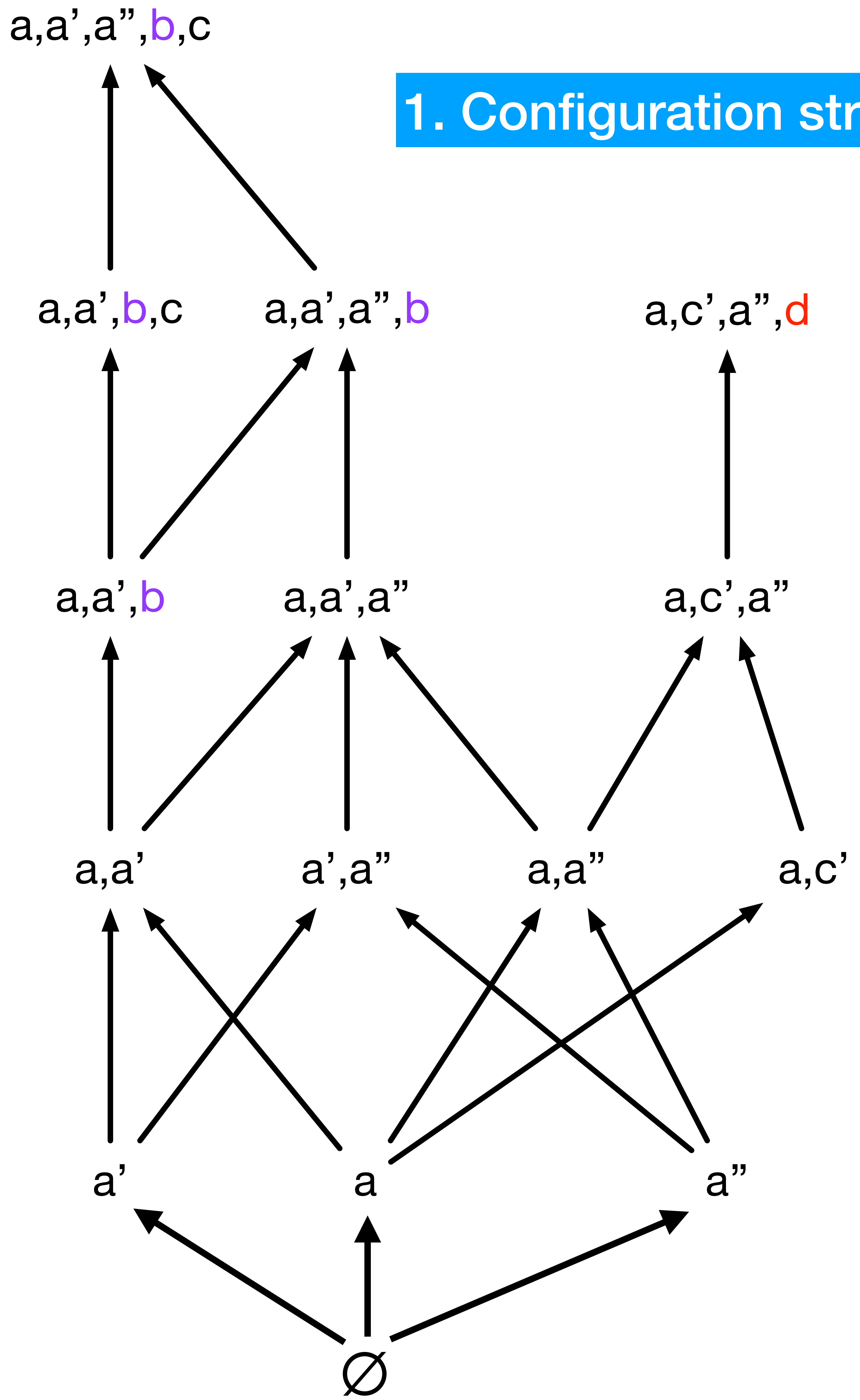
$d_* < \{a''_*, c'_*\}$

$c'_* < a' \quad c'_* < a_*$

$a' < b \wedge b < c$

Overview

1. Configuration structures



Logical characterisation

Time

$\emptyset$

$$\{a, a'\} < b \ \wedge b < c$$

$$a' \# c' \quad a < c'$$

$$\{a'', c'\} < d$$

Time

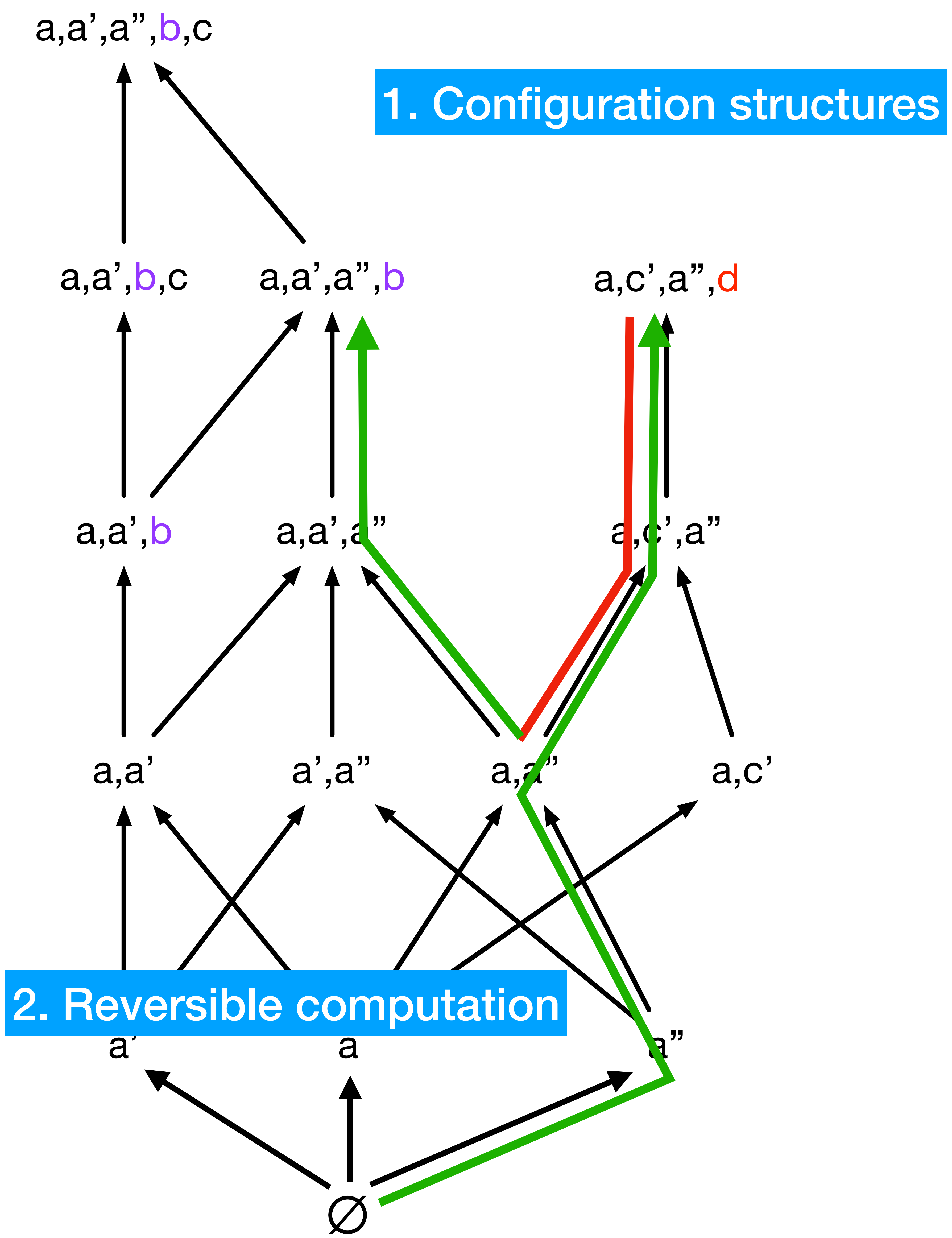
$$\begin{array}{c}\{a, c', a'', d\} \\ \uparrow \\ \emptyset\end{array}$$

$$d_* < \{a''_*, c'_*\}$$

$$c'_* < a' \quad c'_* < a_*$$

$$a' < b \wedge b < c$$

Overview



Logical characterisation

Time

$\emptyset$

$\{a, a'\} < b \;\wedge b < c$

$a' \# c' \quad a < c'$

$\{a'', c'\} < d$

Time

$$\emptyset \longrightarrow \{a, c', a'', d\}$$

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$c'_* < a' \quad c'_* < a_*$

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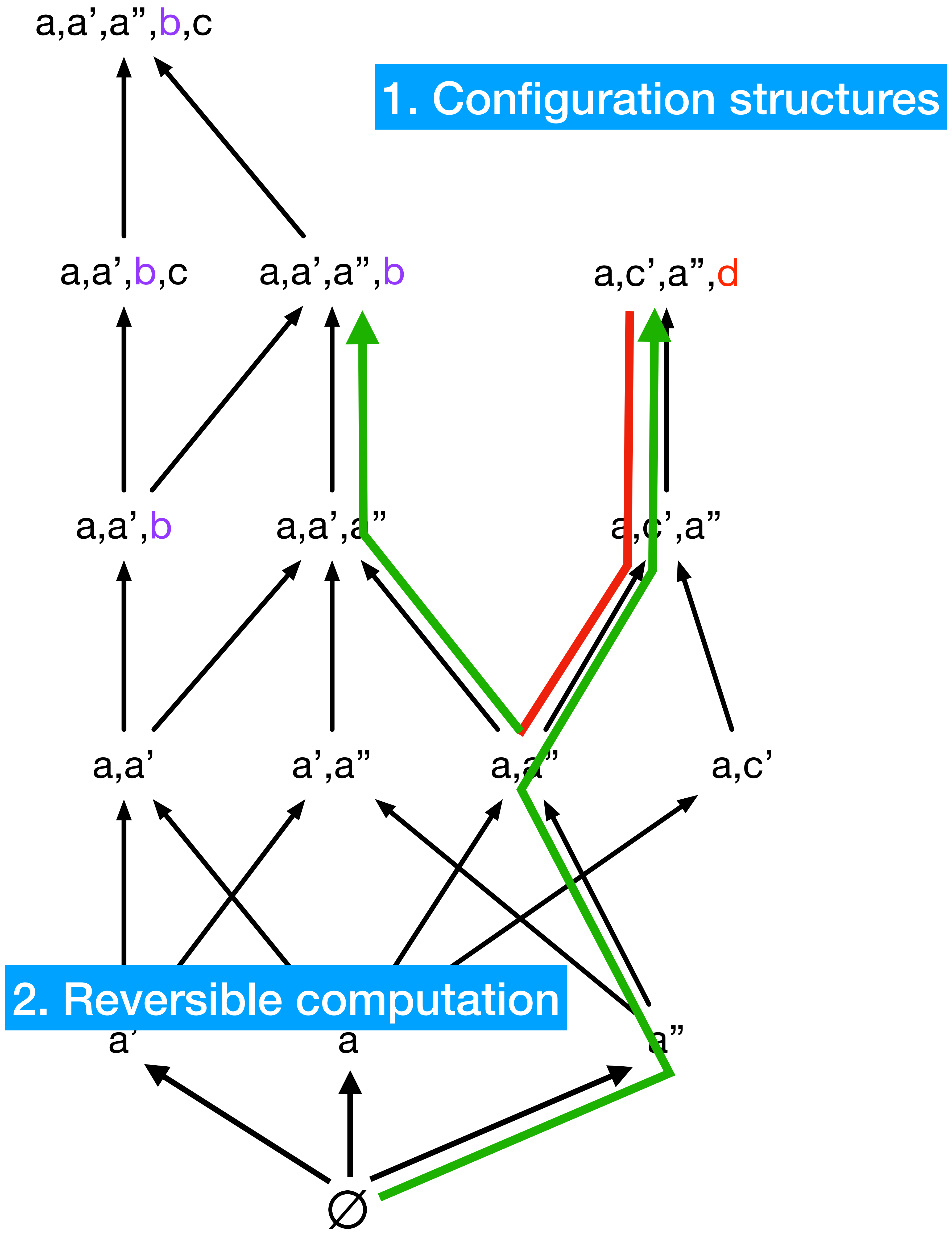
Overview



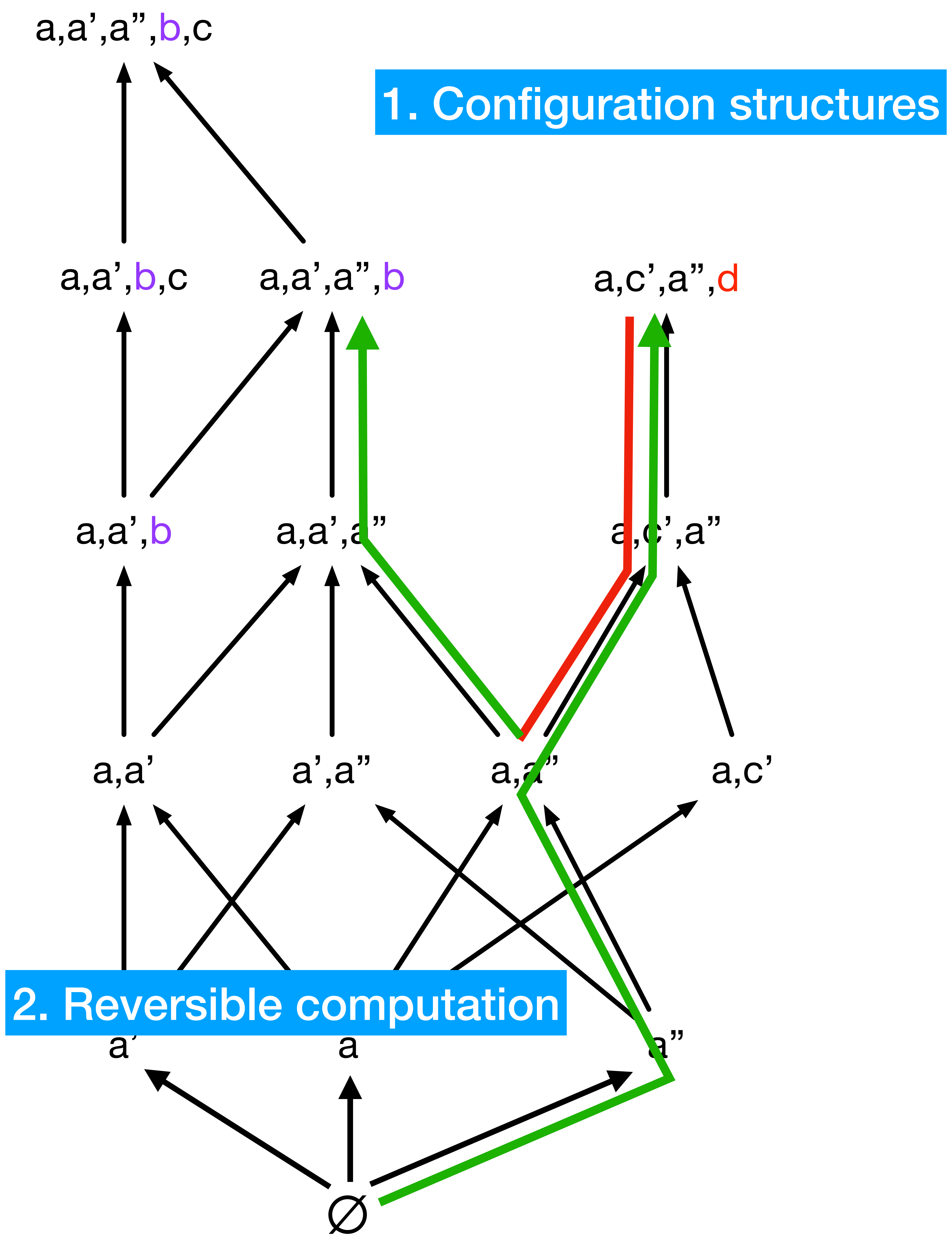
adjunction

Logical characterisation

Time	$\{a, a'\} < b \ \wedge b < c$
$\emptyset$	$a' \ \# \ c' \quad a < c'$
	$\{a'', c'\} < d$

Time	$d_* < \{a''_*, c'_*\}$
$\{a, c', a'', d\}$	$c'_* < a' \quad c'_* < a_*$
$\uparrow$	
$\emptyset$	$a' < b \wedge b < c$

Overview



adjunction

Logical characterisation

Time

$\emptyset$

$\{a, a'\} < b \ \wedge b < c$

$a' \ \# \ c'$

$a < c'$

$\{a'', c'\} < d$

Time

$\{a, c', a'', d\}$

$\uparrow$

$\emptyset$

$d_* < \{a''_*, c'_*\}$

$c'_* < a'$

$c'_* < a_*$

$a' < b \wedge b < c$

3. Prime event structures interpretation

Configuration structure

$$\mathbb{C} = (E, X) \quad X \subseteq \mathscr{P}(E)$$

Partial order

$$\mathbb{C}^{\text{p.o}} = (X, \subseteq\,) \quad x \frown_{\mathbb{C}} y \text{ iff } \uparrow^{\mathbb{C}} \{x, y\} \neq \varnothing$$

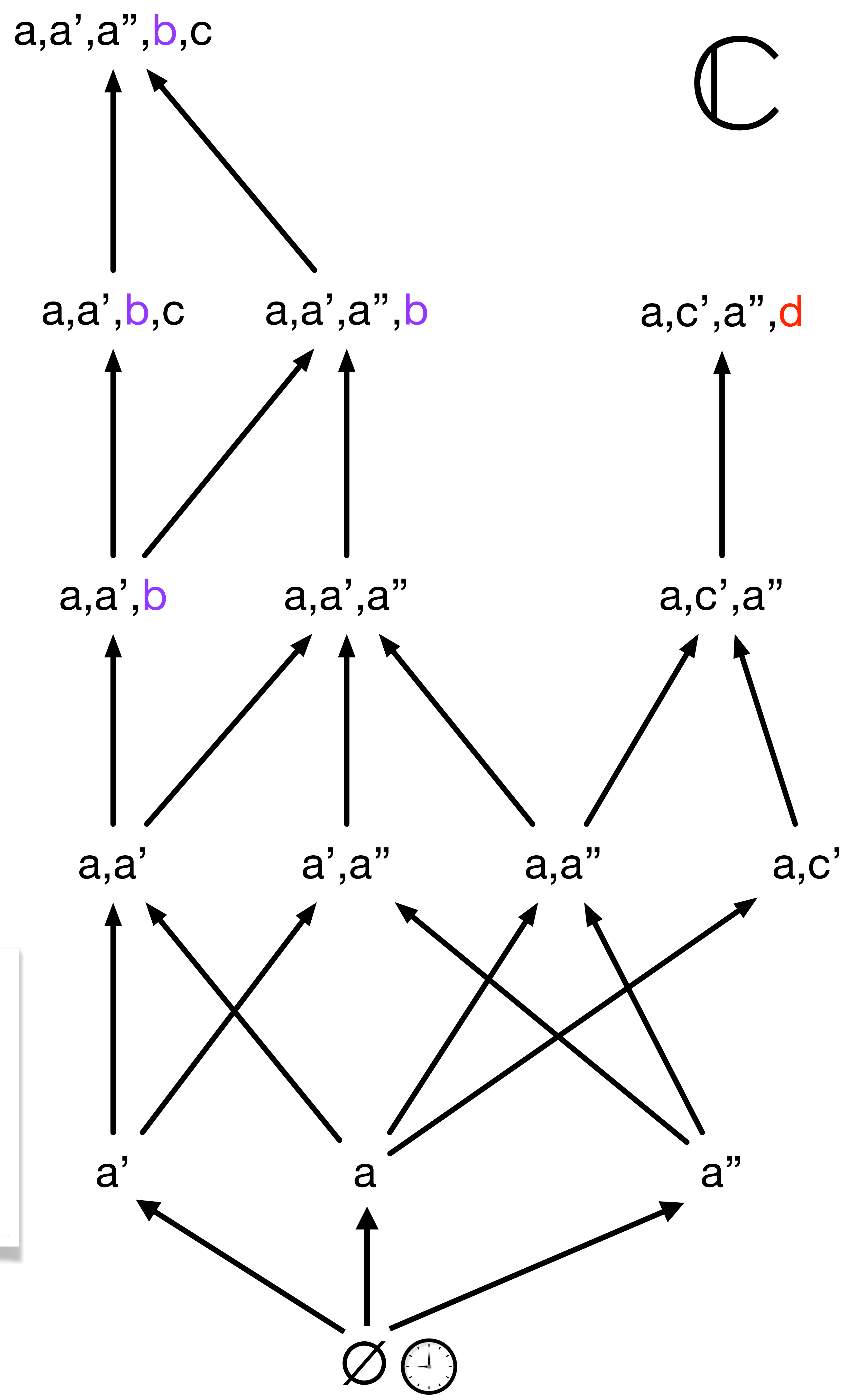
Atomic

$\mathbb{C}$ is *rooted* and *connected*

Definition 2.3 Let $\mathbb{C} \in \mathcal{C}_E$. For all finite $x \in \mathbb{C}$, we define the *residual* of $\mathbb{C}$ *after* x:

$$x \cdot \mathbb{C} := \langle E, \{z \in \mathcal{P}(E) \mid \exists y \in \uparrow^{\mathbb{C}}\{x\} : z = y \backslash x\}\rangle$$

where $y \backslash x := \{a \in y \mid a \notin x\}$ is the classical set difference.



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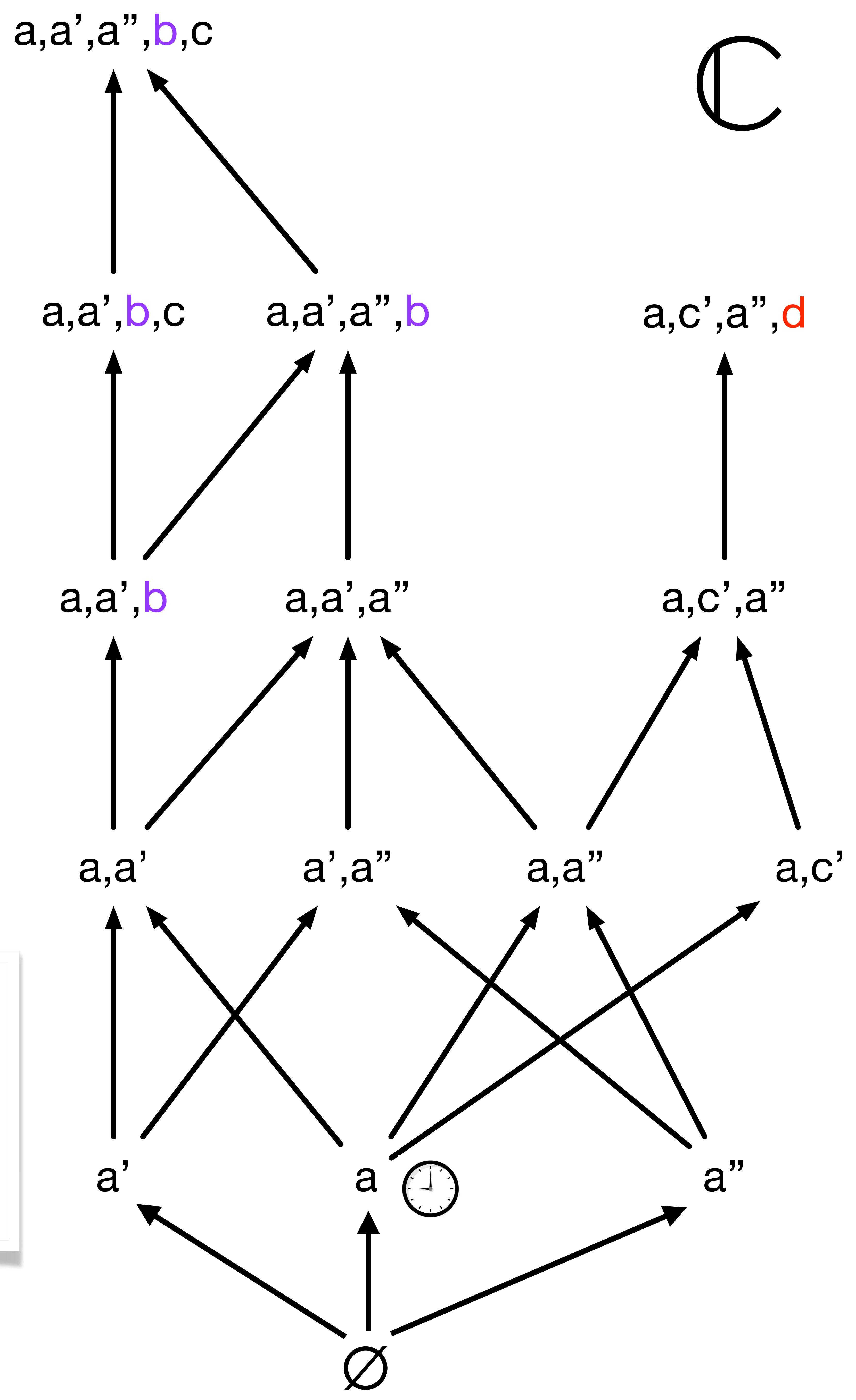

$$\{a\} \cdot \mathbb{C}$$

Configuration structure

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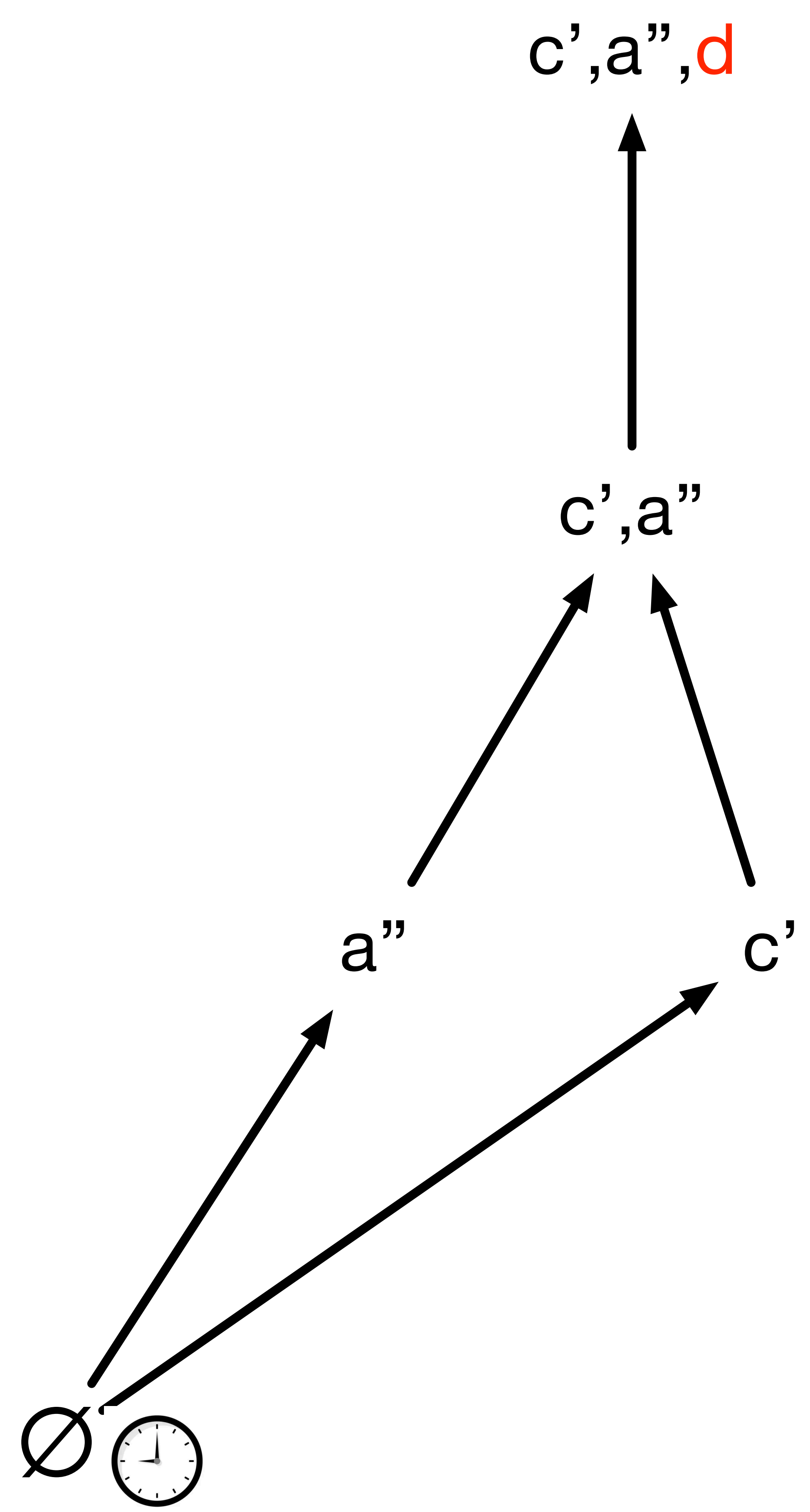

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Forward computation

$$\begin{tikzcd}
 & z & & t \\
x \arrow[ur, "\subseteq"] & & y \arrow[ul, "\subseteq"'] \arrow[ur, "\subseteq"] & \\
 & \emptyset \arrow[ul, "\subseteq"] \arrow[ur, "\subseteq"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & z\backslash x \cdot x \cdot \mathbb{C} = z\backslash y \cdot y \cdot \mathbb{C} & & t\backslash y \cdot y \cdot \mathbb{C} \\
x \cdot \mathbb{C} \arrow[ur, "z\backslash x"] & & y \cdot \mathbb{C} \arrow[ul, "z\backslash y"'] \arrow[ur, "t\backslash y"] & \\
 & \emptyset \cdot \mathbb{C} \arrow[ul, "x"] \arrow[ur, "y"'] & &
\end{tikzcd}$$

$\mathbb{C}$

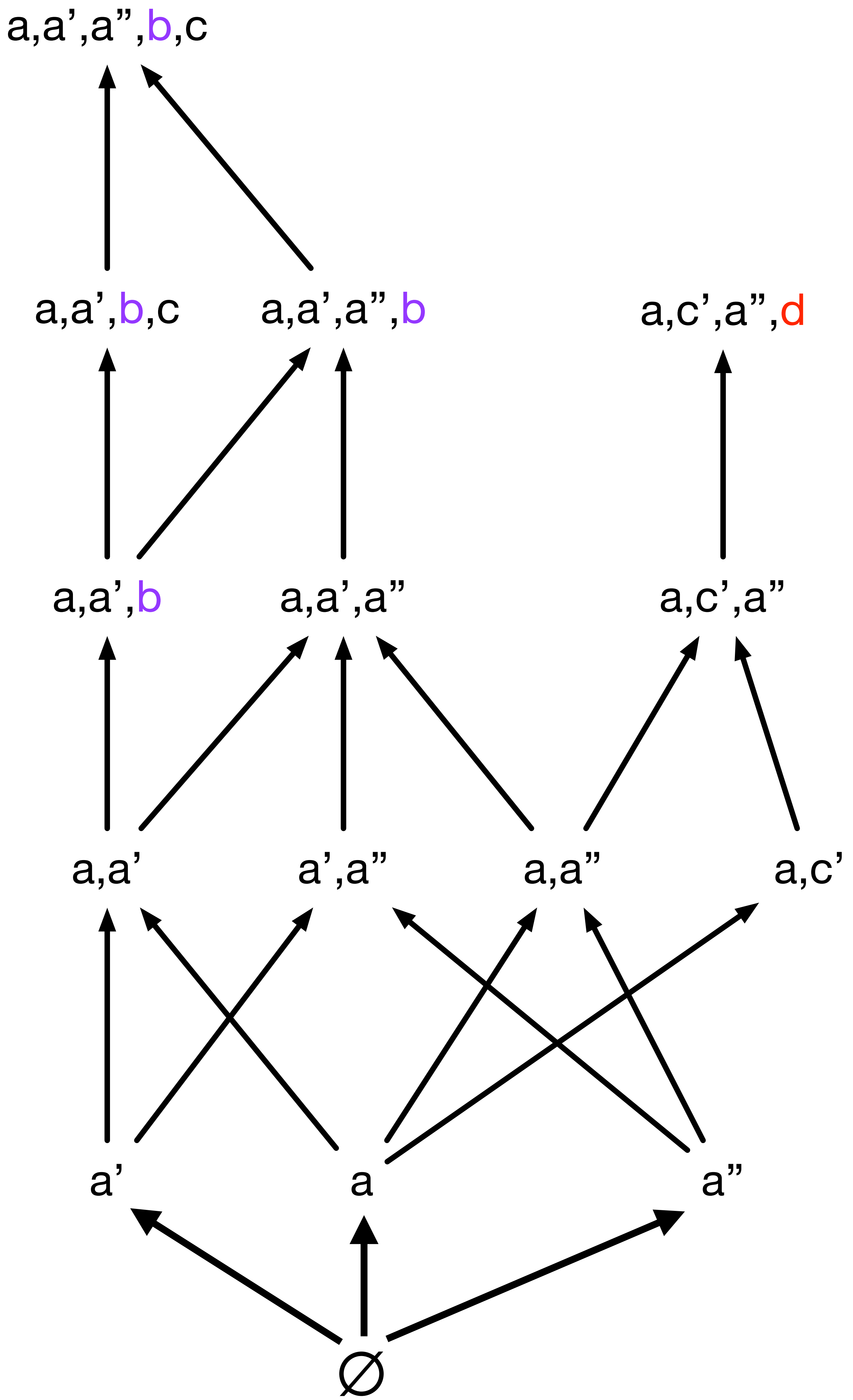
+ residuation

$$x_0 \cdot x_1 \cdot \mathbb{C} = (x_0 \cup x_1) \cdot \mathbb{C}$$

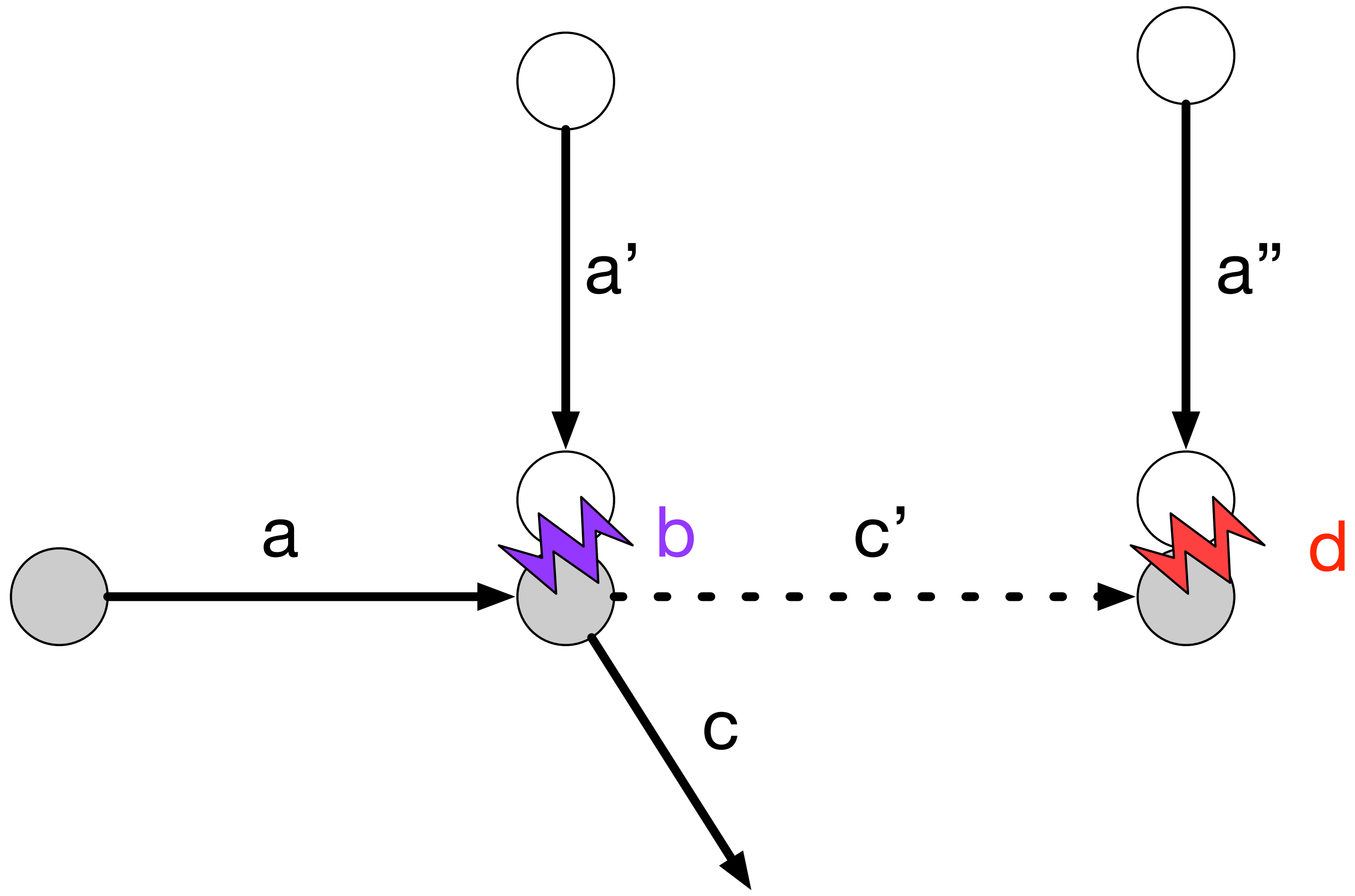
$$\emptyset \cdot \mathbb{C} = \mathbb{C}$$

Labelled transition system
as a monoid action

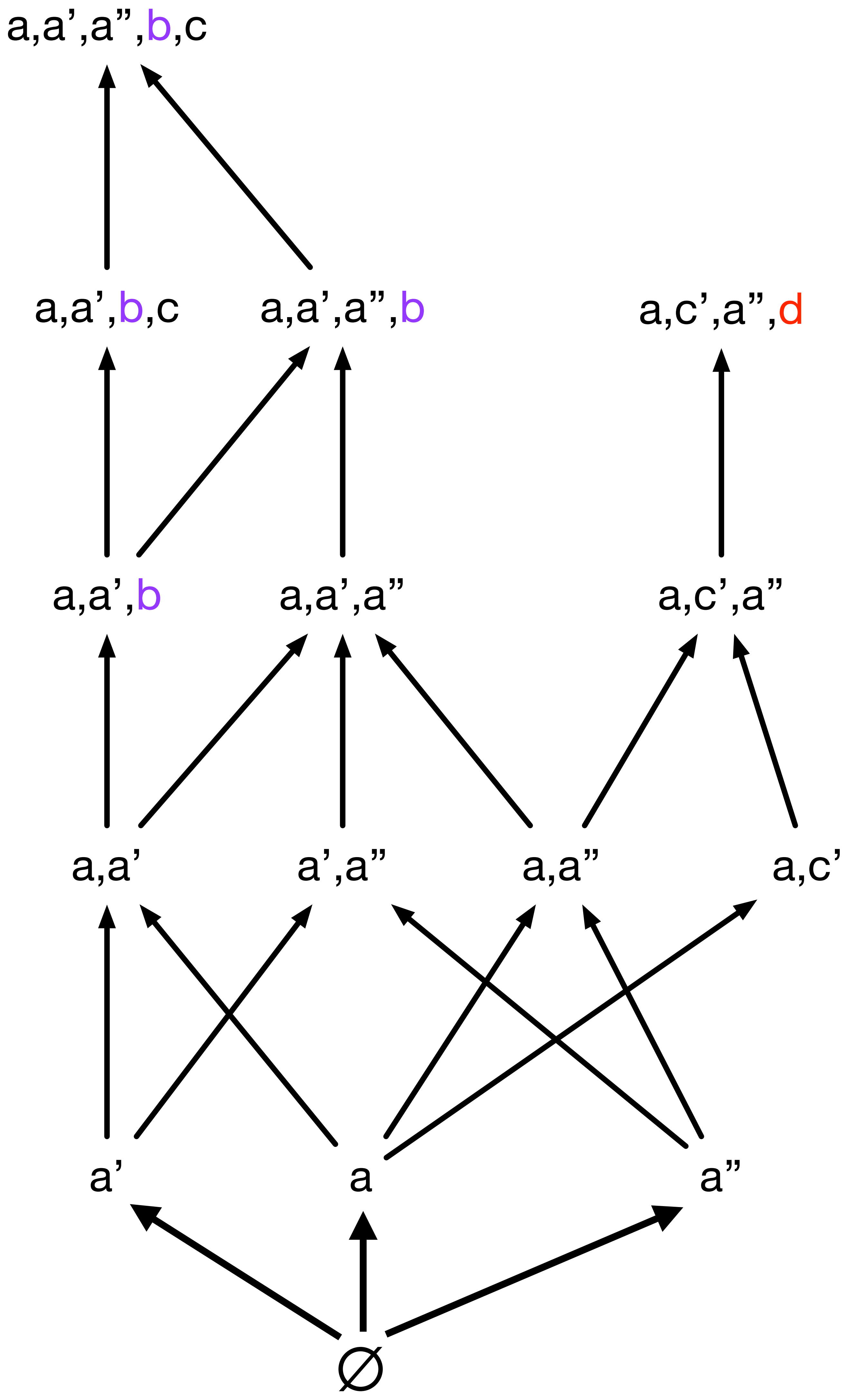
Prime Event Structures



Is this structure engendered by a causality and conflict relation amongst events?



Prime Event Structures



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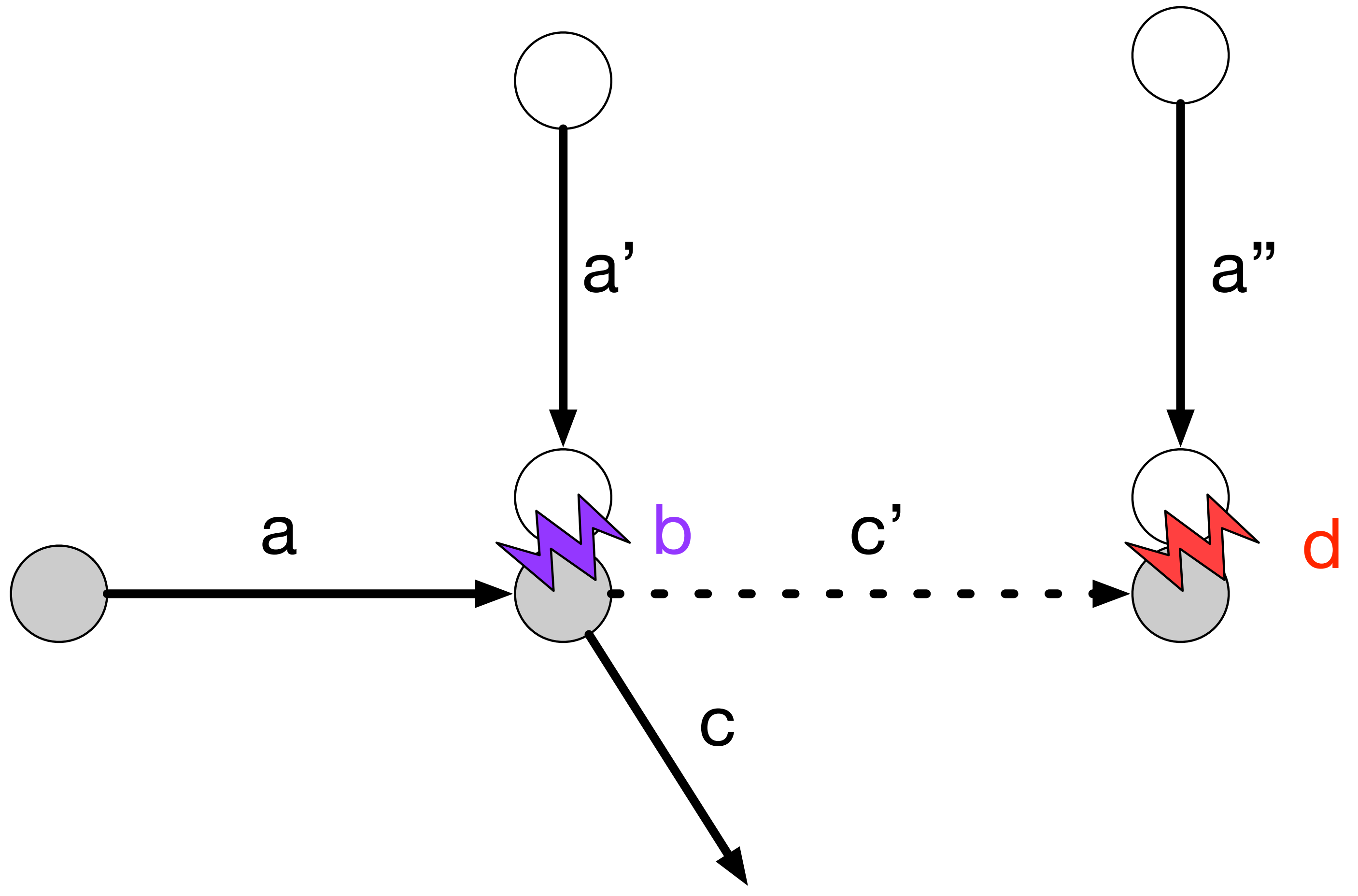

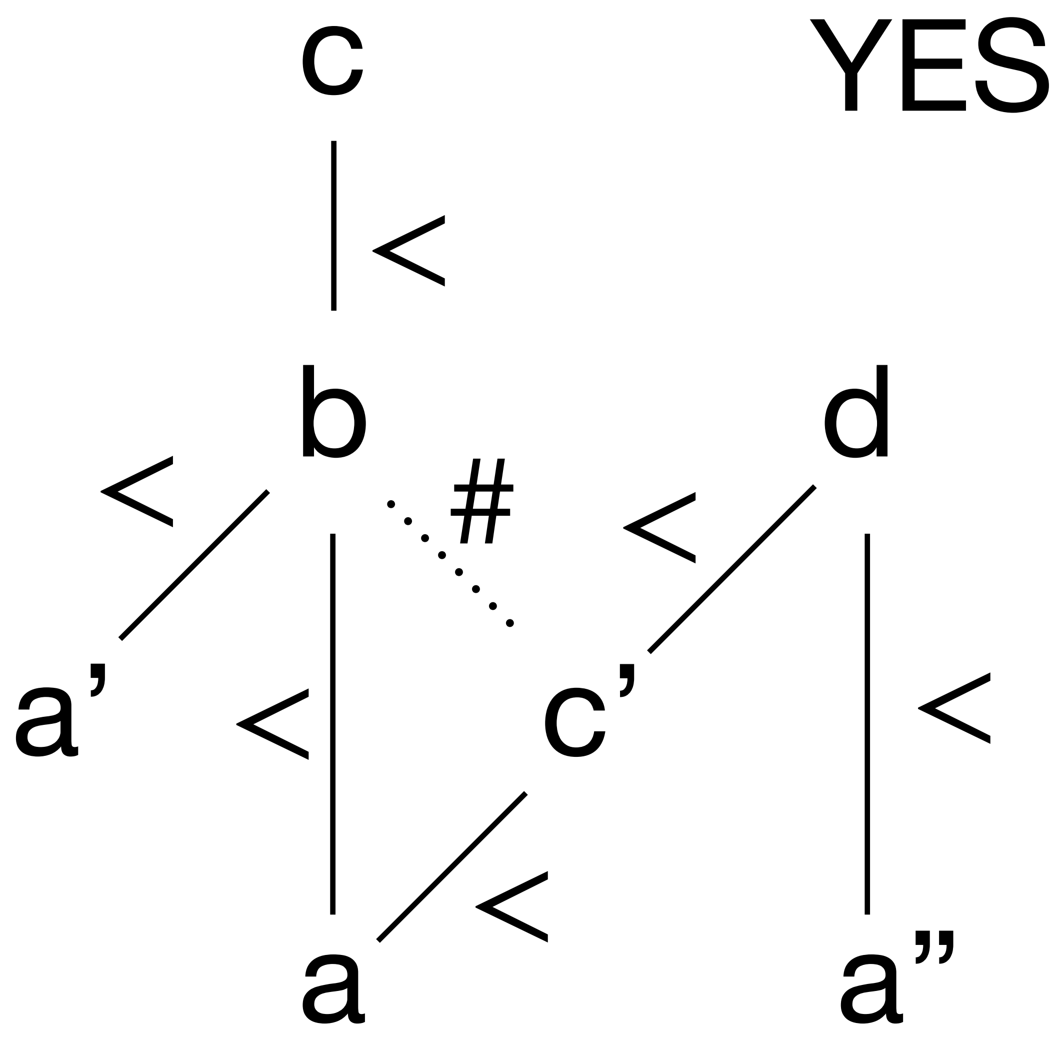

YES!

… but it is not true in general…

Adjunction [Winskel 82]

$$\begin{tikzcd}
 & \{a,b\} & & \{b,c\} \\
\{a\} \arrow[ur] & & \{b\} \arrow[ul] \arrow[ur] & \\
 & \emptyset \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

$\mathbb{C}$

Adjunction [Winskel 82]

Prime elements:

$$p \in \mathsf{Pr}(\mathbb{C}) : p \sqsubseteq \bigsqcup X \implies p \sqsubseteq x \in X$$

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\{a\} \arrow[ur] & & \{b\} \arrow[ul] \arrow[ur] & \\
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\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{b,c\} \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & \\
\{a\} \arrow[rr, dash, dotted] & & \{b\}
\end{tikzcd}$$

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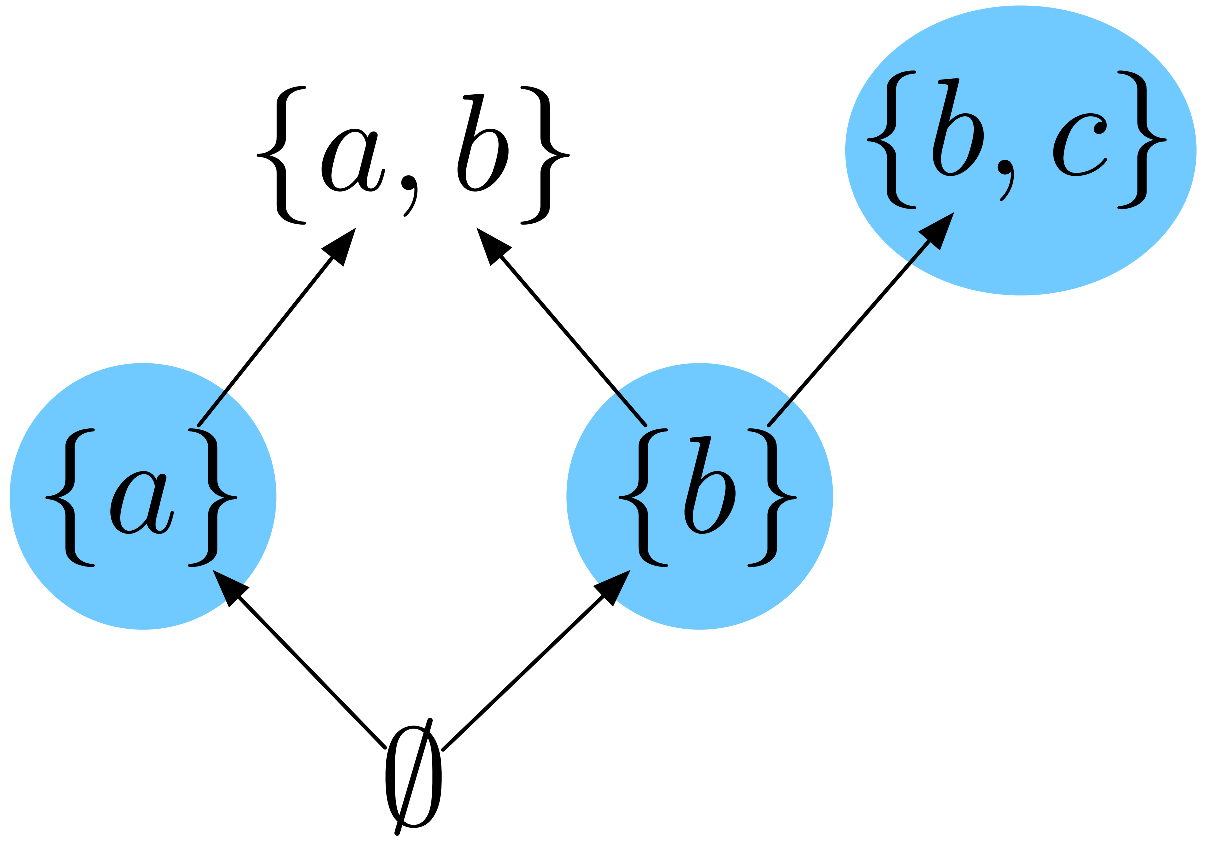
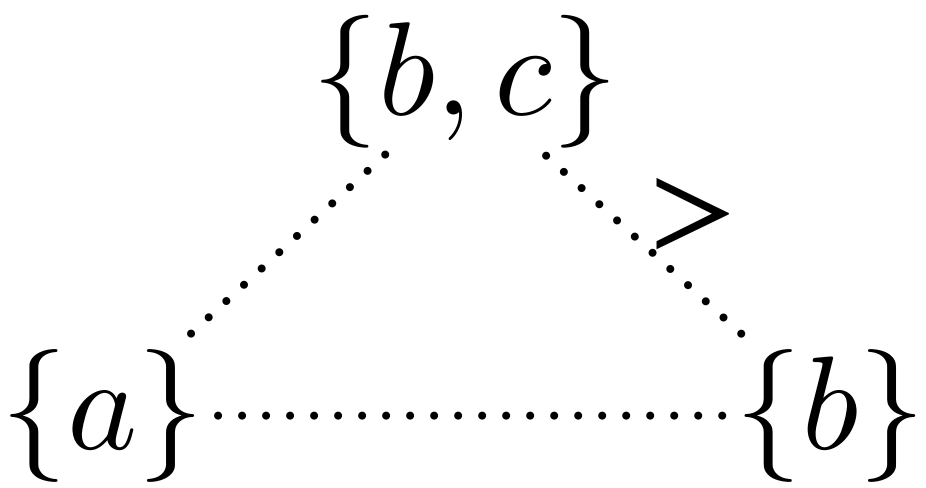


$\mathbb{C}$



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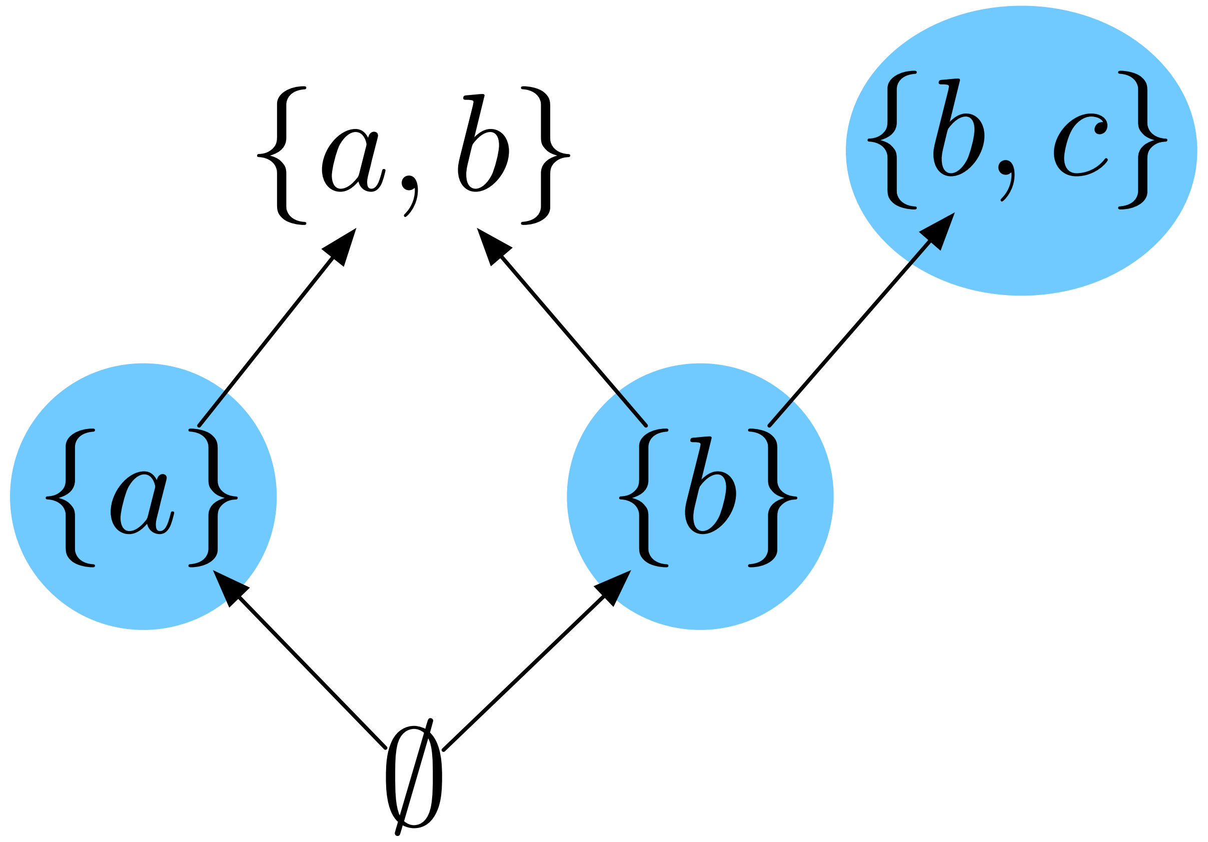
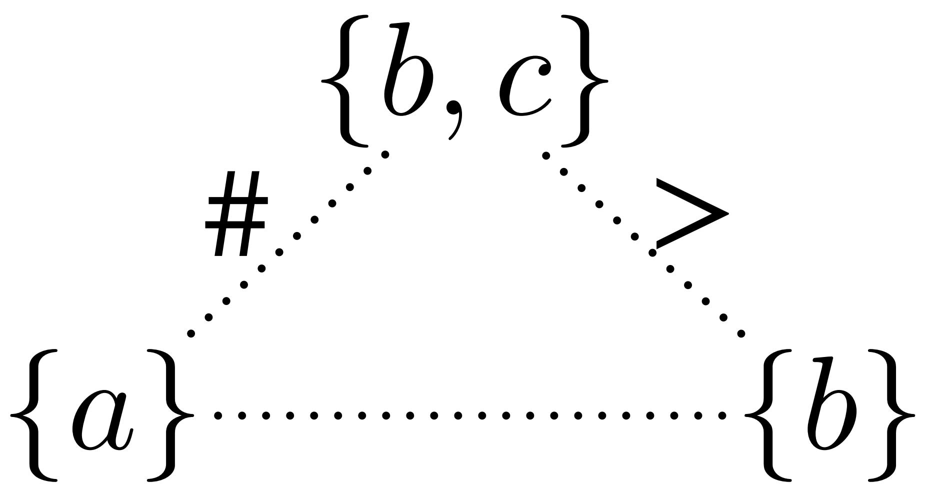


$\mathbb{C}$



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\end{tikzcd}$$

$\mathbb{C}$

$$\begin{tikzcd}
 & \{b,c\} \arrow[dl, dash, dotted, "\#"'] \arrow[dr, dash, dotted, ">"] & \\
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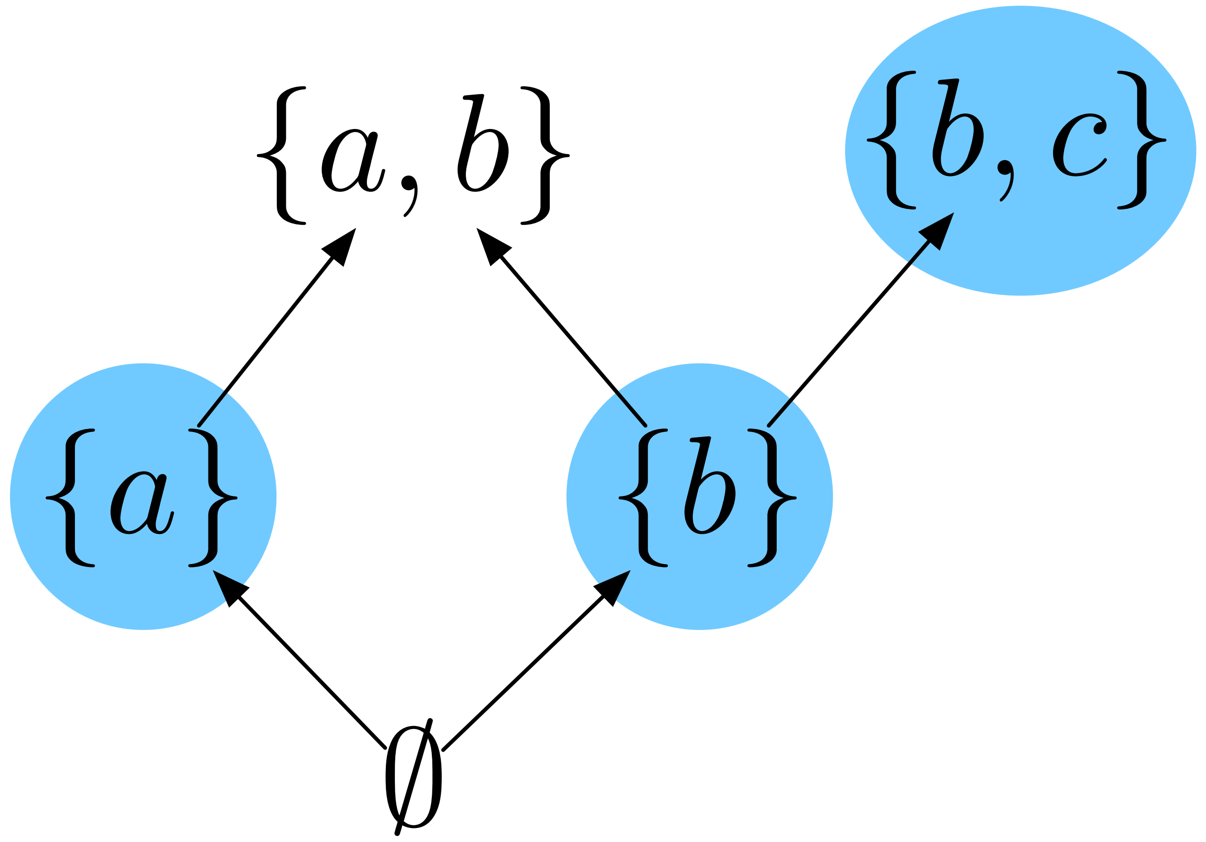
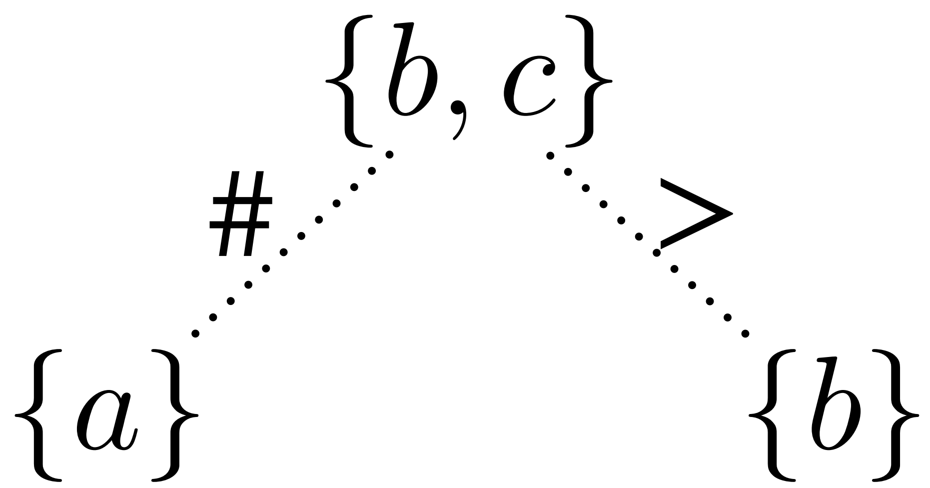


$\mathbb{C}$



$\mathbb{E} = (\mathsf{Pr}(\mathbb{C}), <, \#)$

Adjunction [Winskel 82]

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& \emptyset \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

$\mathbb{C}$

$$\begin{tikzcd}
& \{b, c\} \arrow[dl, dotted, no head, "\#"'] \arrow[dr, dotted, no head, ">"] & \\
\{a\} & & \{b\}
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$\mathbb{C}' = \mathsf{Conf}(\mathbb{E})$

Adjunction [Winskel 82]

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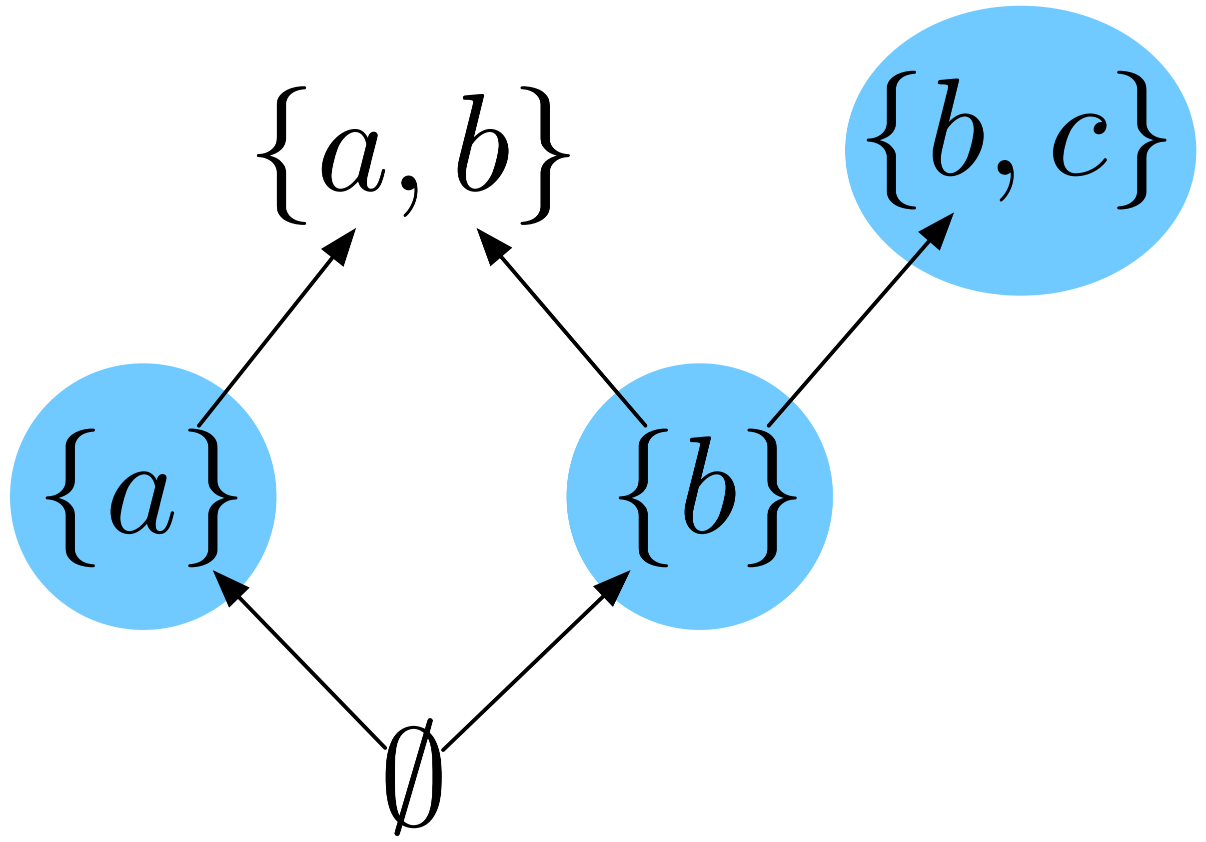
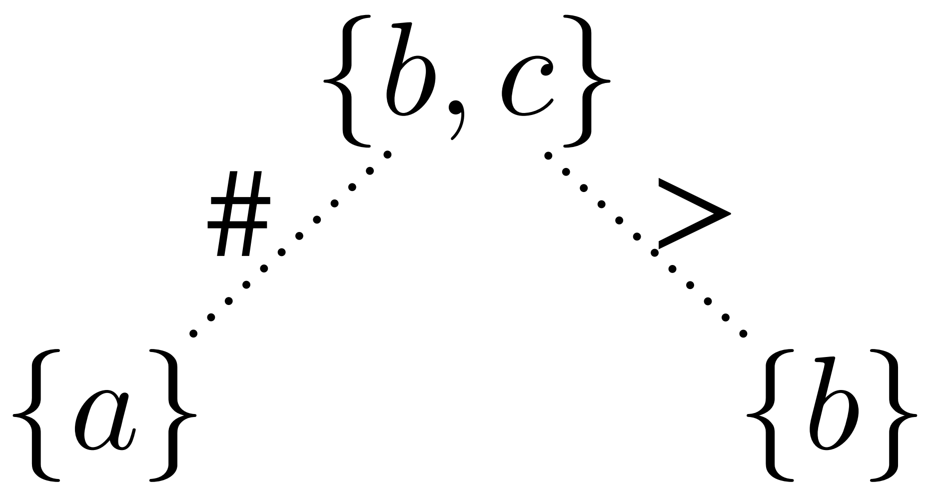


$\mathbb{C}$ prime algebraic & coherent



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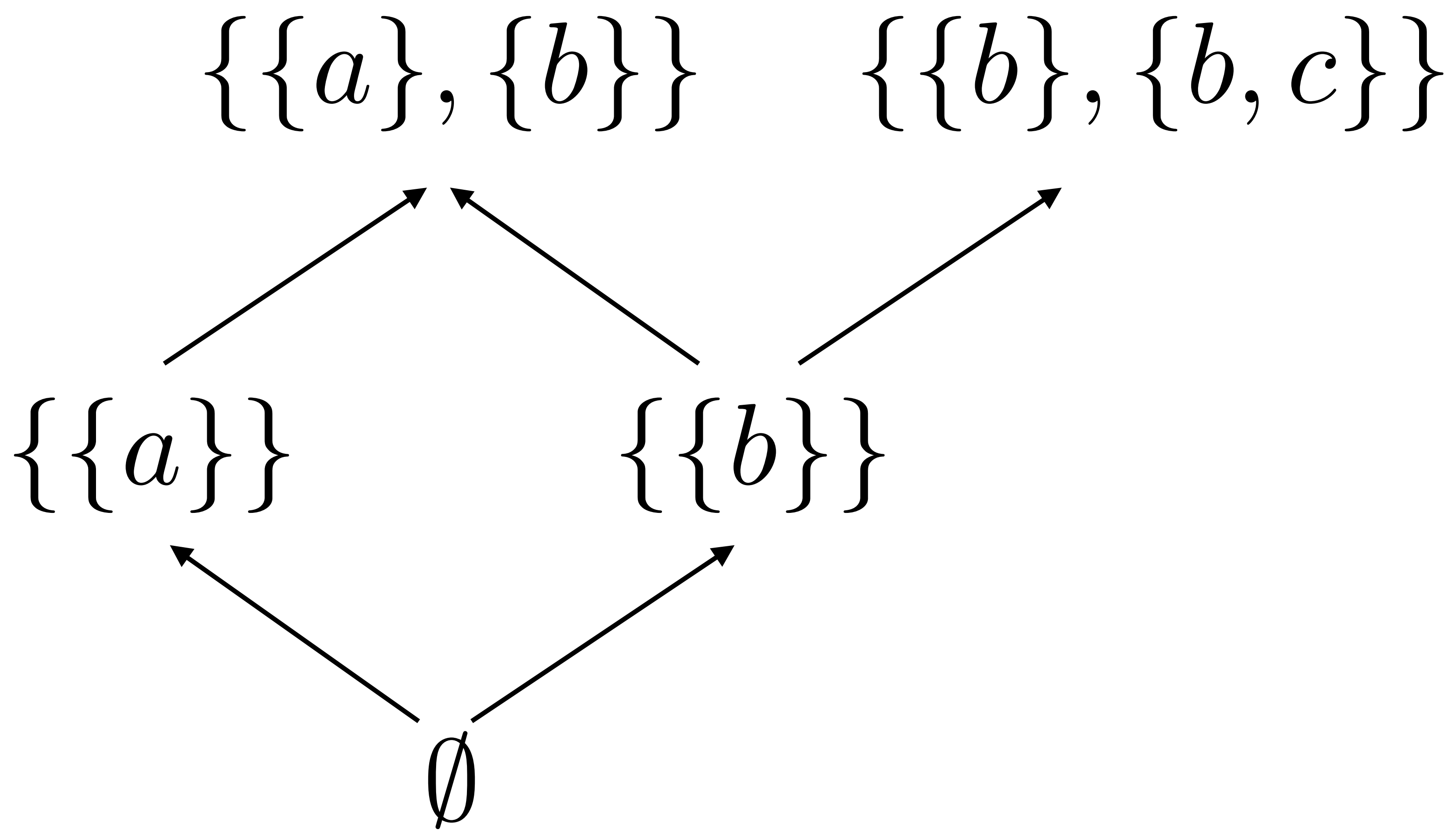

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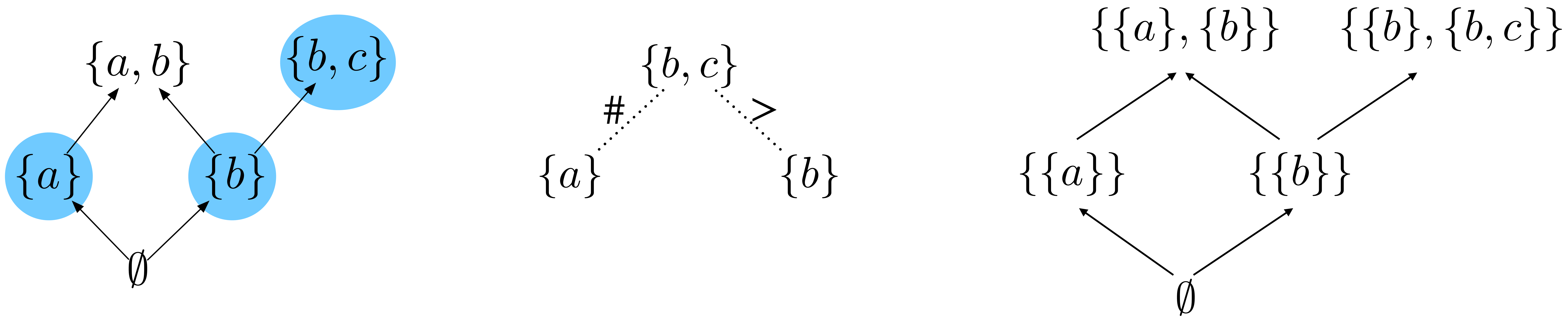


prime algebraic
& coherent

$\mathbb{C}$

$\mathbb{E} = (\mathrm{Pr}(\mathbb{C}), <, \#)$

$\mathbb{C}' = \mathrm{Conf}(\mathbb{E})$

$\simeq$

Stable configuration structures

$$
\begin{tikzcd}
\{a,b\} & & \{a,c\} & & \{b,c\} \\
\{a\} \arrow[u] \arrow[urr] & & \{b\} \arrow[ull] \arrow[urr] & & \{c\} \arrow[ull] \arrow[u] \\
& & \emptyset \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}
$$

$\mathbb{C}_0$

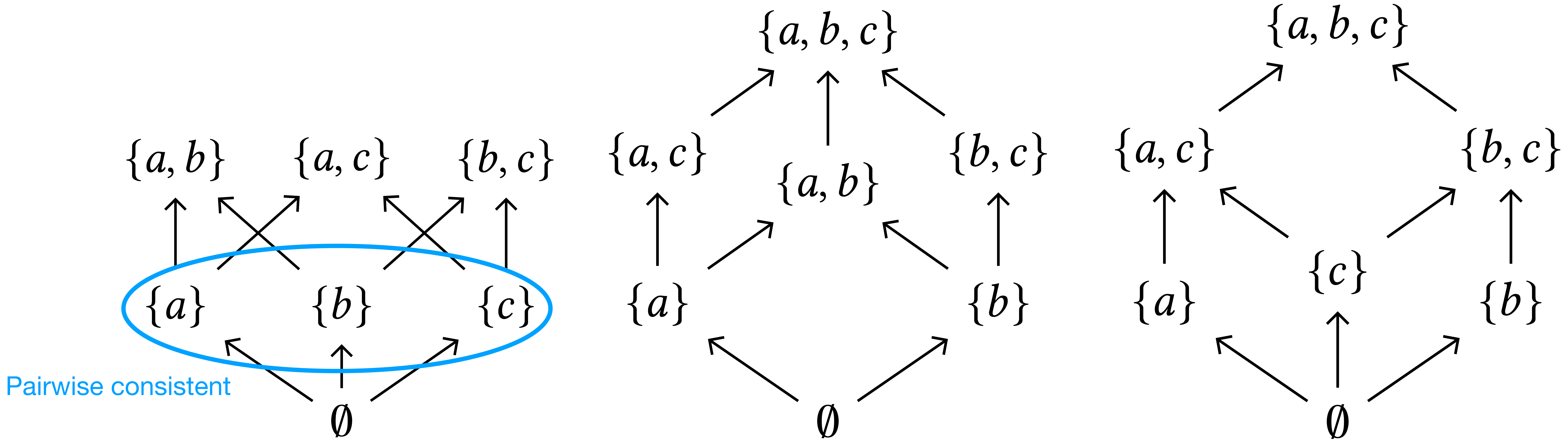
$$
\begin{tikzcd}
& \{a,b,c\} & \\
\{a,c\} \arrow[ur] & \{a,b\} \arrow[u] & \{b,c\} \arrow[ul] \\
\{a\} \arrow[u] \arrow[ur] & & \{b\} \arrow[ul] \arrow[u] \\
& \emptyset \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$\mathbb{C}_1$

$$
\begin{tikzcd}
& \{a,b,c\} & \\
\{a,c\} \arrow[ur] & & \{b,c\} \arrow[ul] \\
\{a\} \arrow[u] & \{c\} \arrow[ul] \arrow[ur] & \{b\} \arrow[u] \\
& \emptyset \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

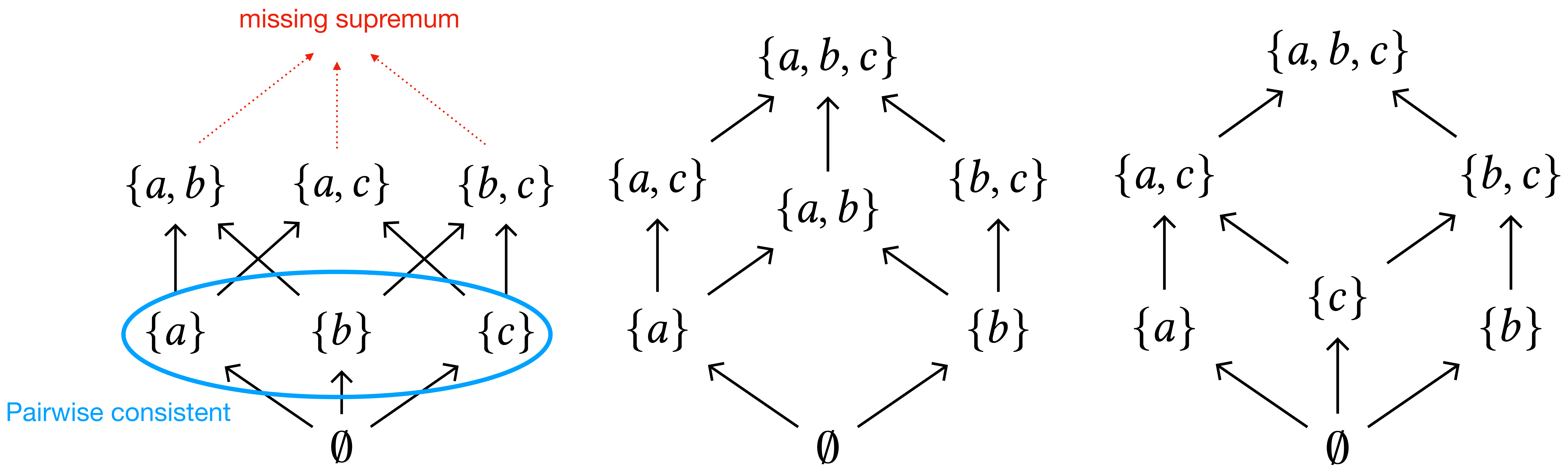
$\mathbb{C}_2$

Stable configuration structures



$\mathbb{C}_0$ $\mathbb{C}_1$ $\mathbb{C}_2$

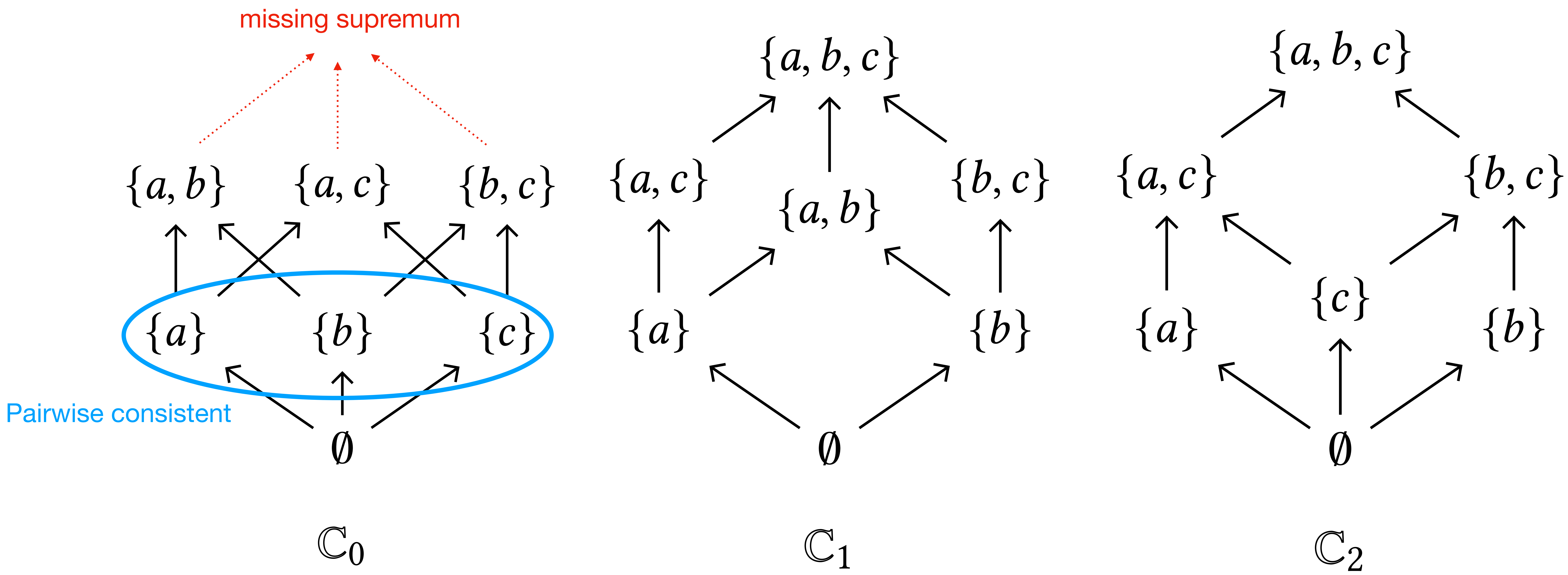
Stable configuration structures



$\mathbb{C}_0$ $\mathbb{C}_1$ $\mathbb{C}_2$

Not coherent

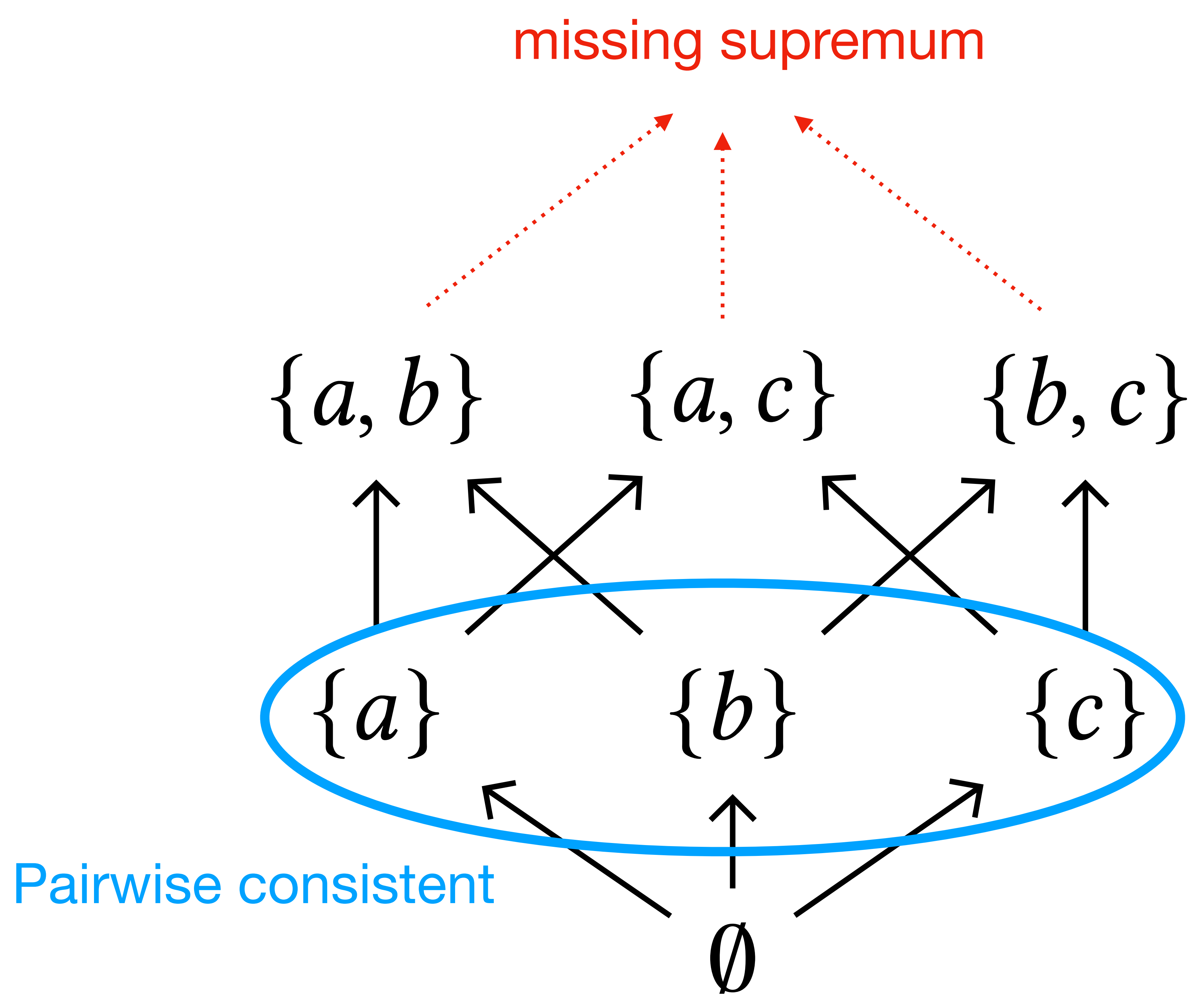
Stable configuration structures



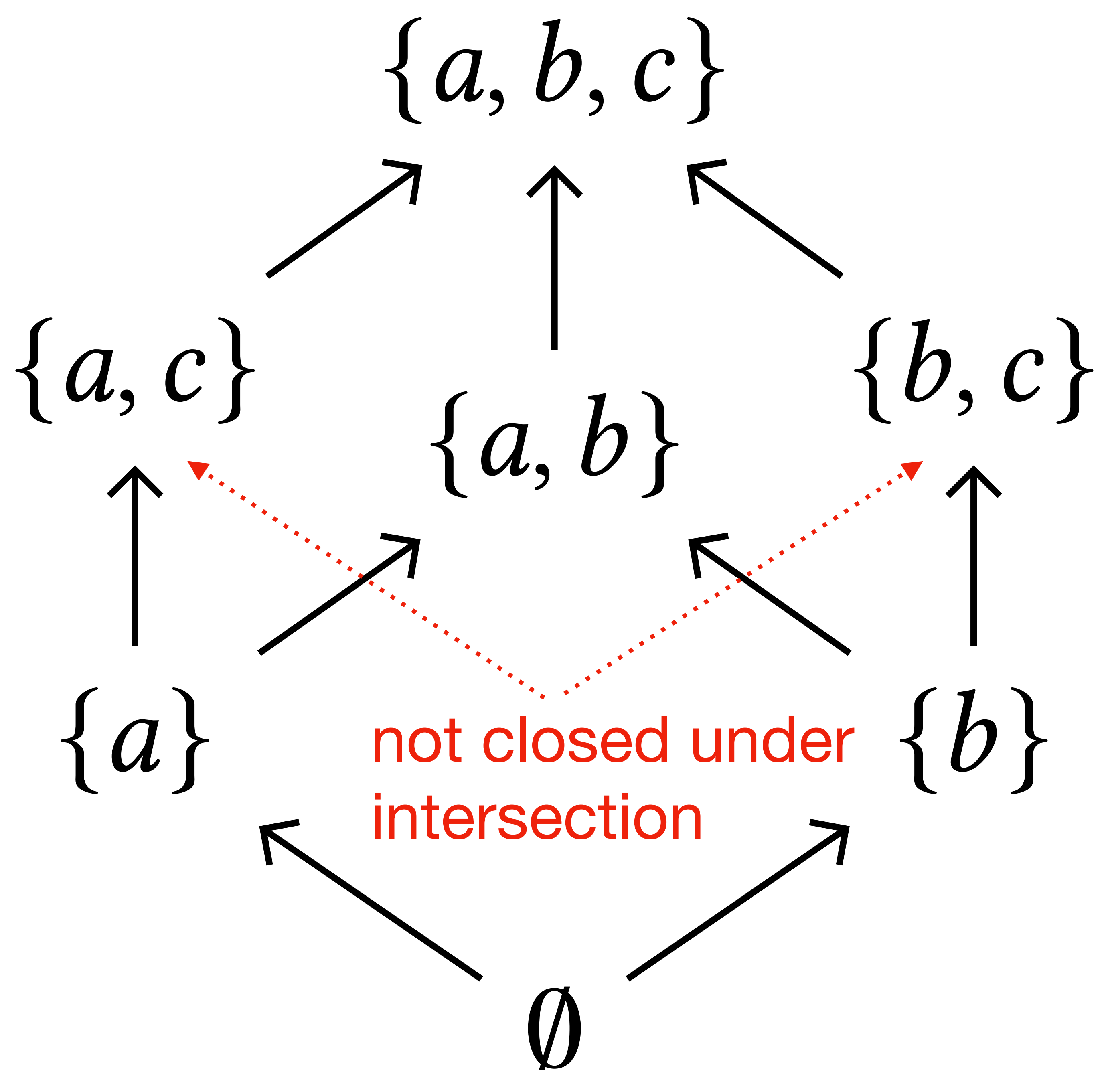
Not coherent

Not prime algebraic

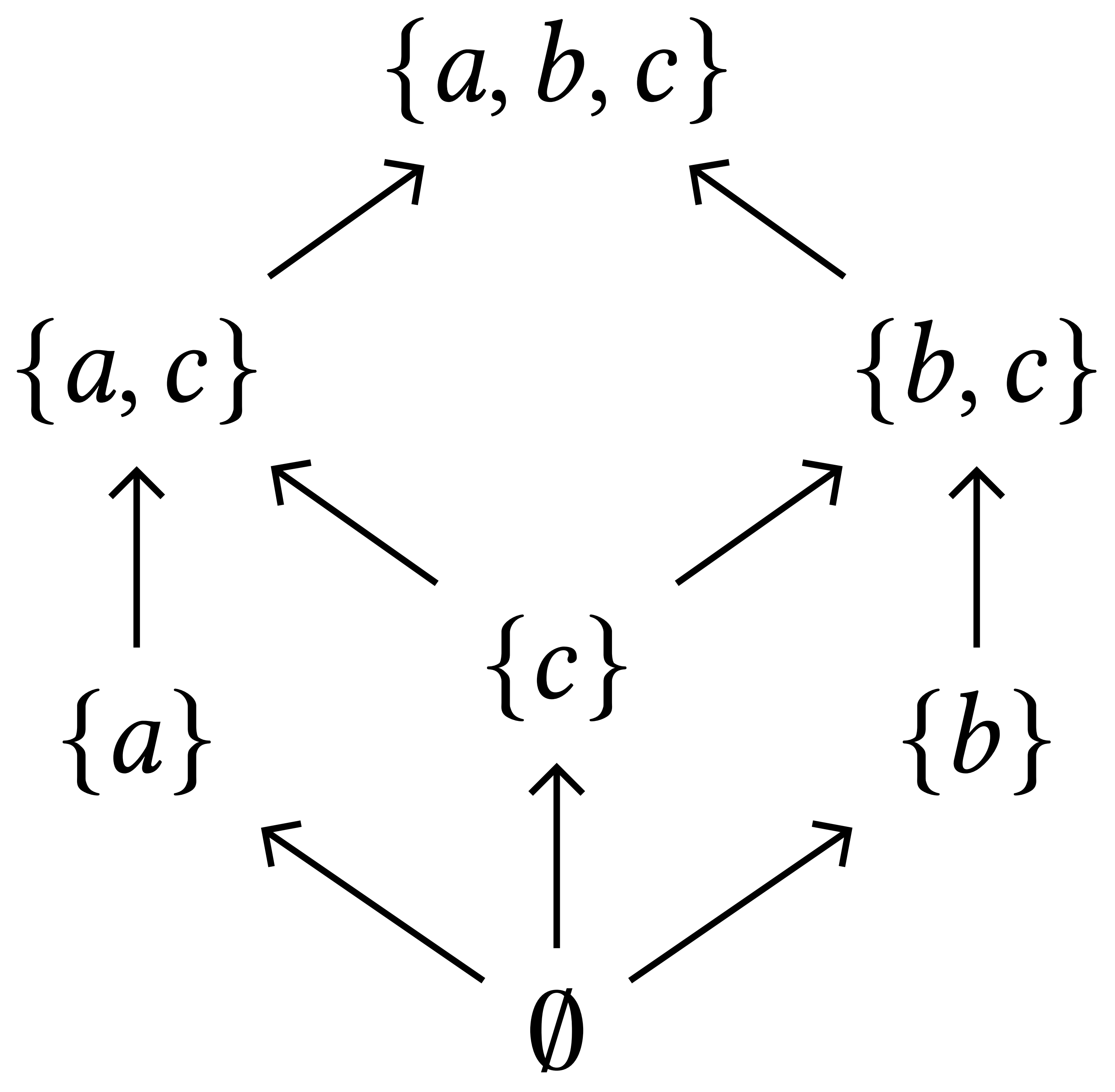
Stable configuration structures



$\mathbb{C}_0$



$\mathbb{C}_1$

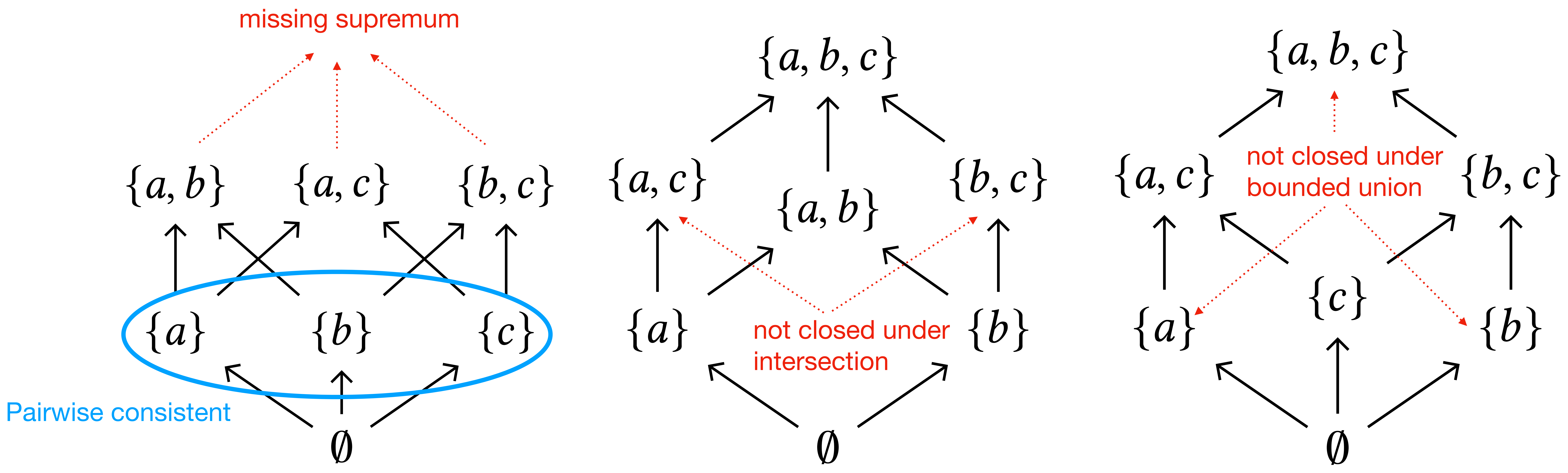


$\mathbb{C}_2$

Not coherent

Not prime algebraic

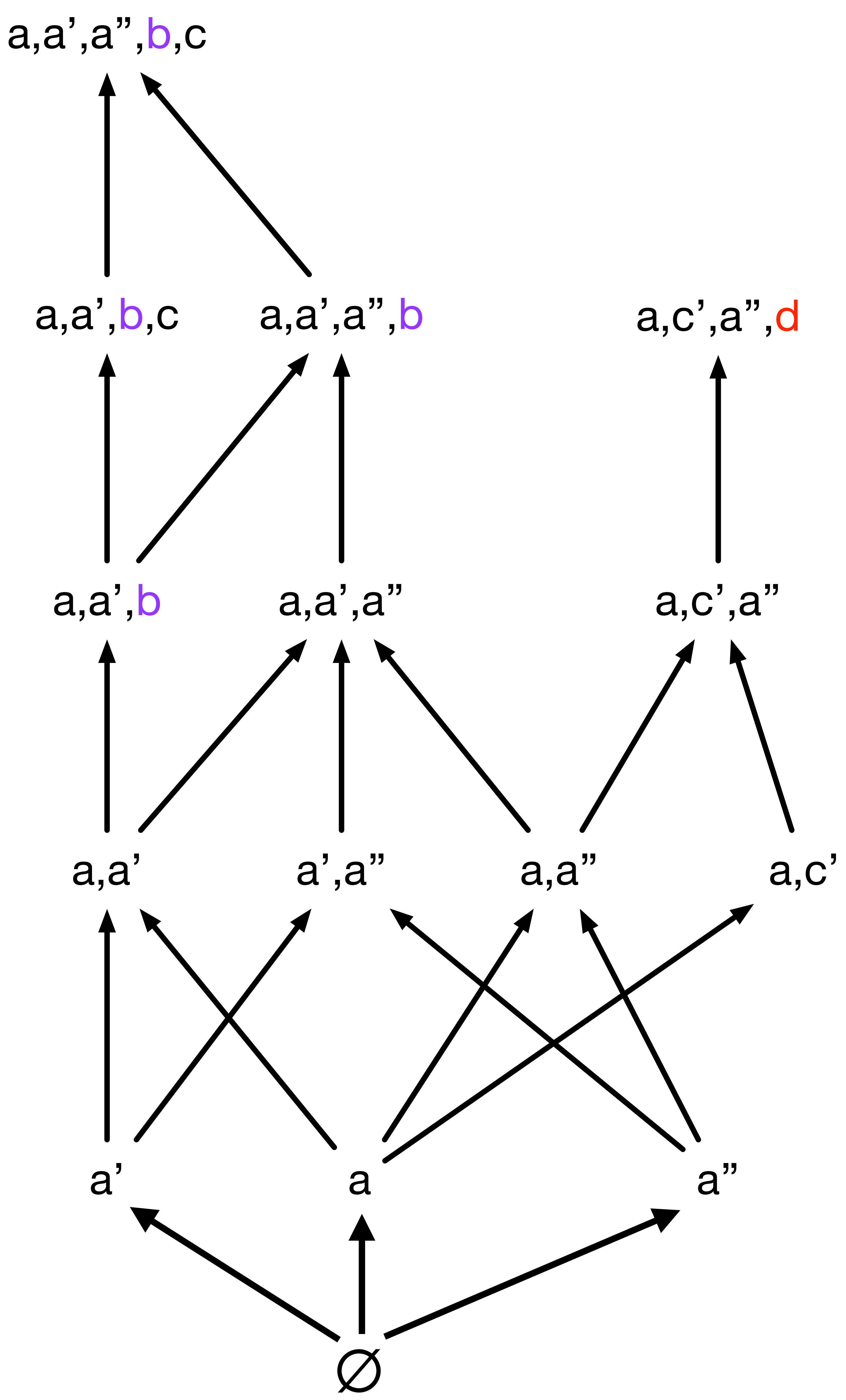
Stable configuration structures



$\mathbb{C}_0$ $\mathbb{C}_1$ $\mathbb{C}_2$

Not coherent

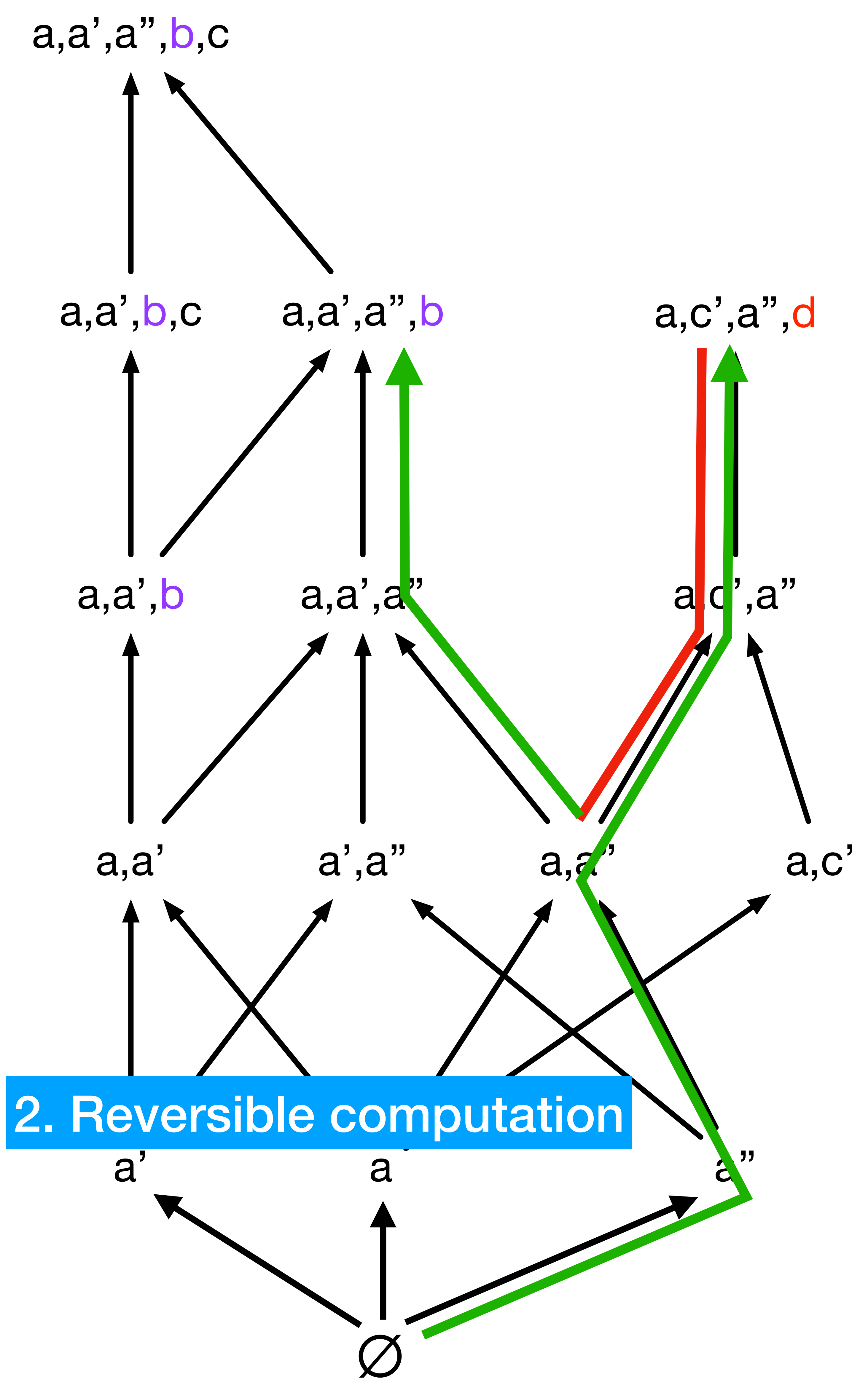
Not prime algebraic



adjunction

Logical characterisation

Time	$\{a, a'\} < b \ \wedge b < c$
$\varnothing$	$a' \ \# \ c' \quad a < c'$
	$\{a'', c'\} < d$



$\Leftrightarrow$

adjunction

Logical characterisation

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	$\{a'', c'\} < d$

Symmetric residual

Definition 2.3 Let $\mathbb{C} \in \mathcal{C}_E$. For all finite $x \in \mathbb{C}$, we define the *residual* of $\mathbb{C}$ *after* x:

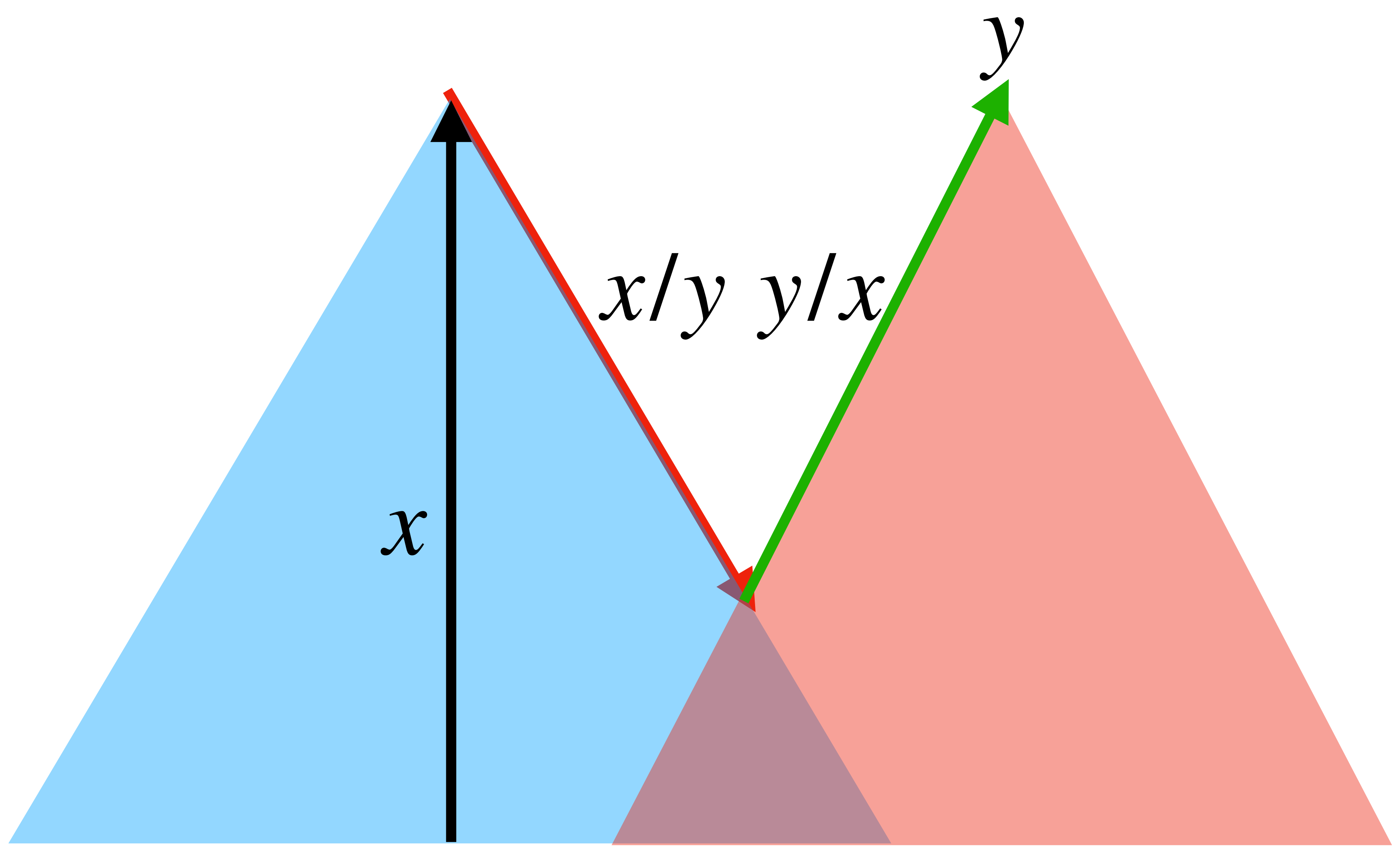
$$x \cdot \mathbb{C} := \langle E, \{z \in \mathcal{P}(E) \mid \exists y \in \uparrow^{\mathbb{C}}\{x\} : z = y \backslash x\}\rangle$$

where $y \backslash x := \{a \in y \mid a \notin x\}$ is the classical set difference.

$x \Delta y =_{def} x \backslash y \cup y \backslash x$

$(\mathscr{P}(E), \Delta)$ is a group

if $x \subseteq y$ then $x \Delta y = y \backslash x$



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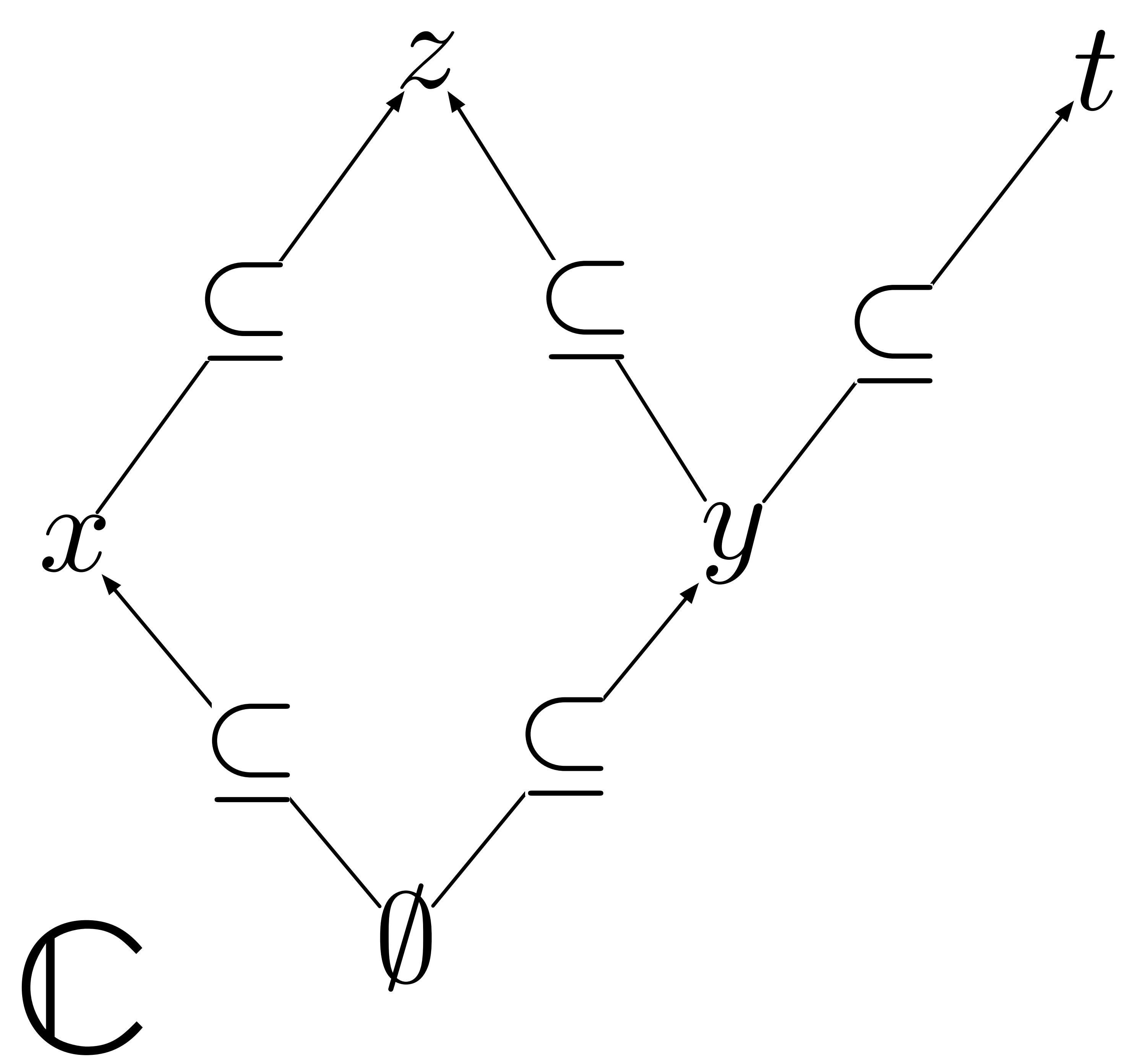
Definition 3.1 Let $\mathbb{C} \in \mathcal{C}_E$. For all finite $x \in \mathbb{C}$, we define the *symmetric residual of* $\mathbb{C}$ *after* x:

$$x \odot \mathbb{C} := \langle E, \{z \in \mathcal{P}(E) \mid \exists y \in \mathbb{C} : z = y \mathbin{\triangle} x\} \rangle.$$

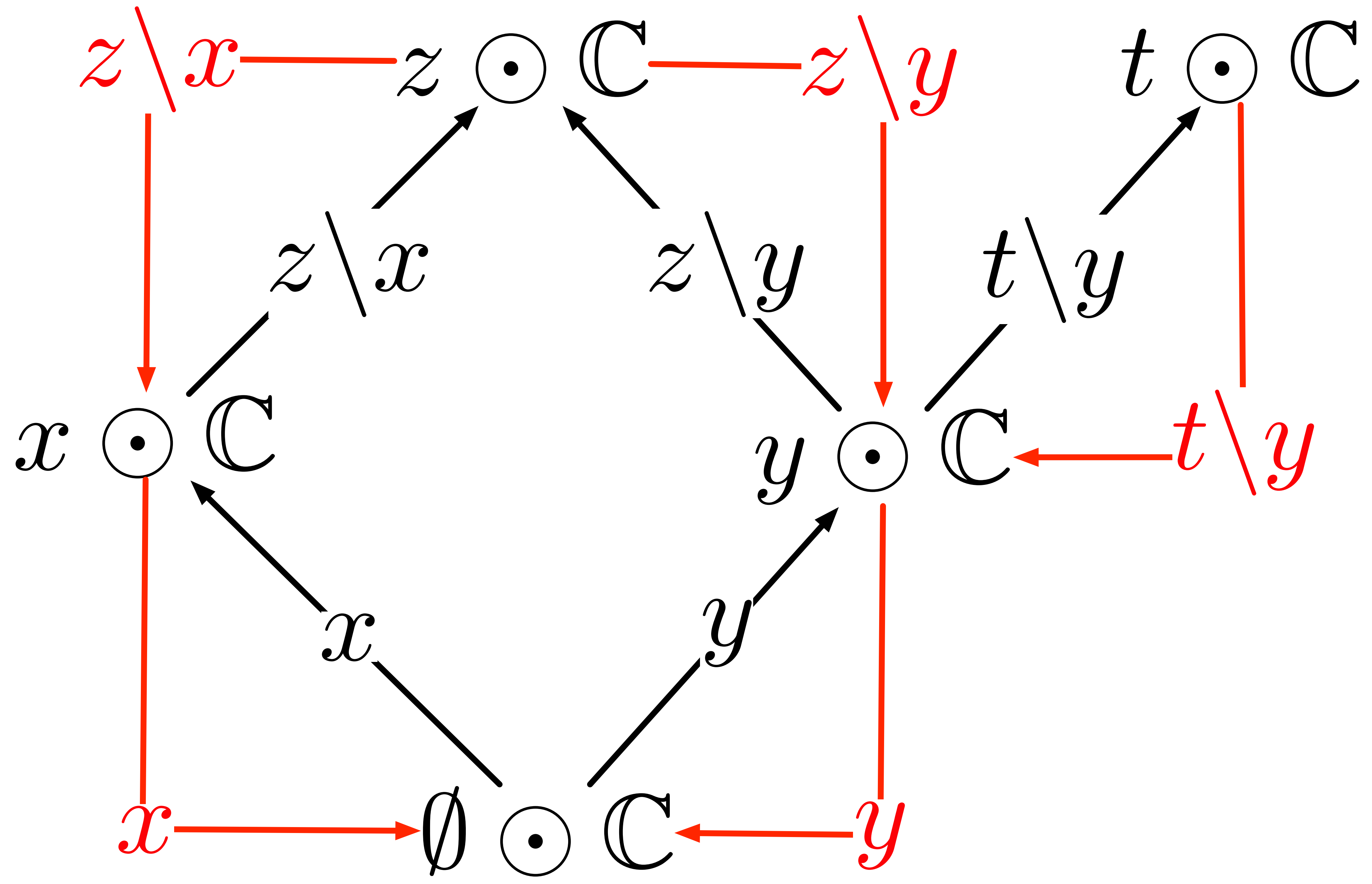
Proposition 3.2 (Group action) *The operator* $(\odot) : \mathcal{P}_{fin}(E) \times \mathcal{C}_E \to \mathcal{C}_E$ *is a* group action *on configuration structures, i.e.:*

- *for all finite configurations* x, y, *if* $x \in \mathbb{C}$ *and* $y \in x \odot \mathbb{C}$, *then* $x \odot (y \odot \mathbb{C}) = (x \mathbin{\triangle} y) \odot \mathbb{C}$.
- $\emptyset \odot \mathbb{C} = \mathbb{C}$.

Reversible computation

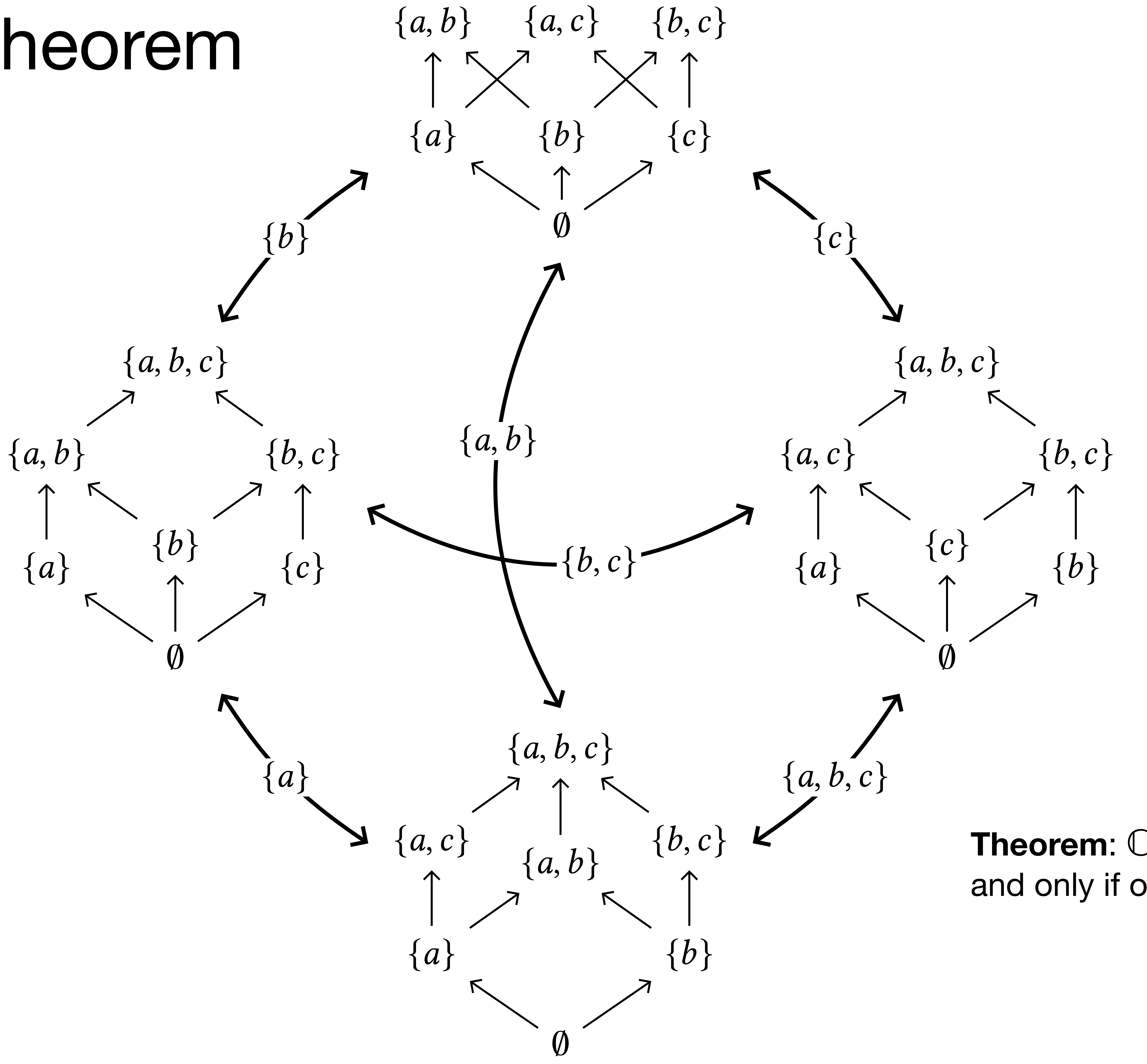


+ symmetric residuation



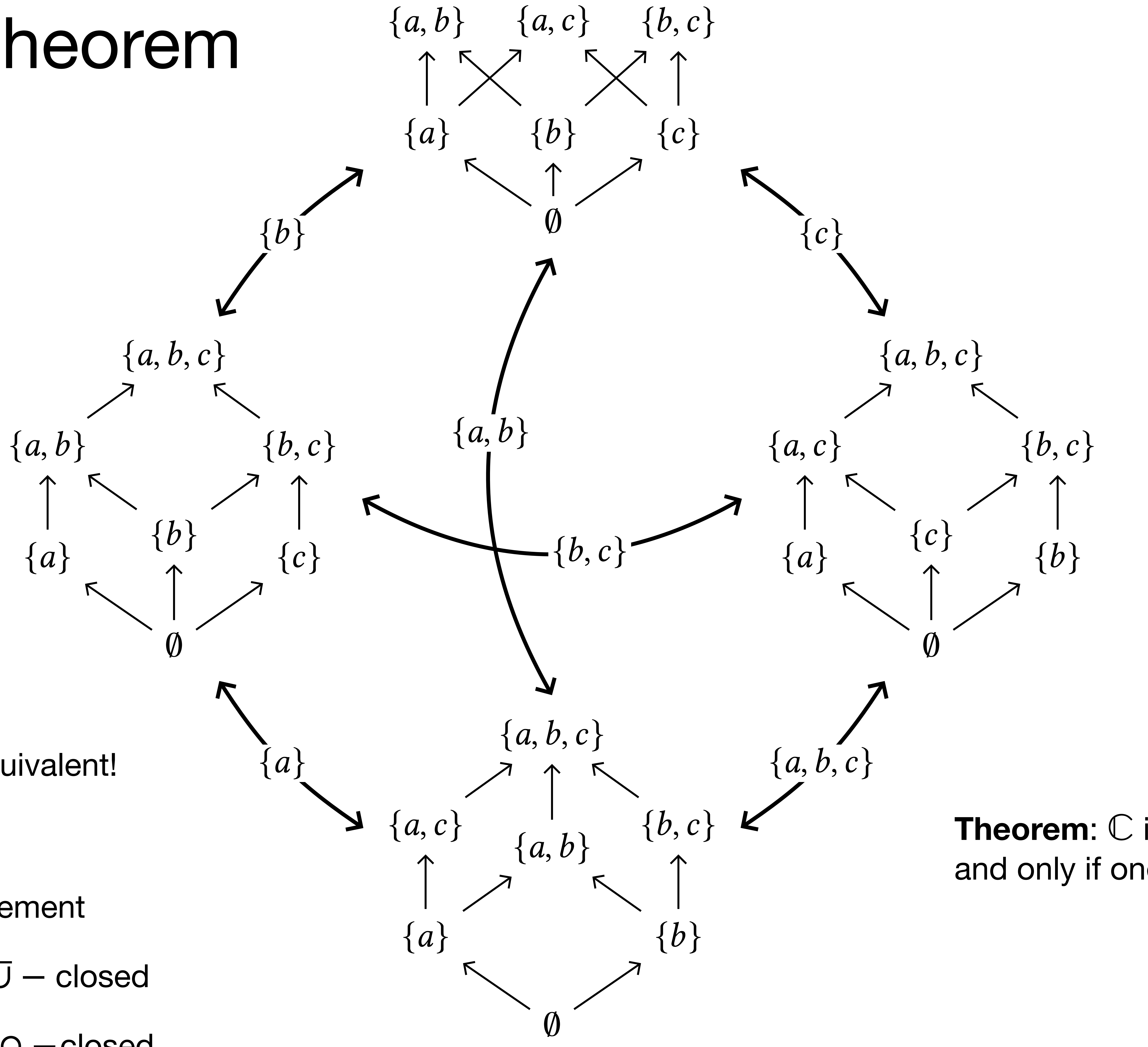
(conservative) labelled transition system as a group action's orbit

Stable orbit theorem



Theorem: $\mathbb{C}$ is a prime event structure if and only if one the point in its orbit is.

Stable orbit theorem



Property: The following are equivalent!

- $\mathbb{C}$ is not an event structure
- $\mathrm{orb}(\mathbb{C})$ has an incoherent element
- $\mathrm{orb}(\mathbb{C})$ has an element not $\overline{\cup}-$closed
- $\mathrm{orb}(\mathbb{C})$ has an element not $\cap-$closed

Theorem: $\mathbb{C}$ is a prime event structure if and only if one the point in its orbit is.

Interpreting symmetric residual

p.a + coh

$$\begin{tikzcd} \mathbb{C} \arrow[d, shift right] \arrow[d, phantom, "\sim"] \\ (\mathrm{Pr}(\mathbb{C}), <, \#) \arrow[u, shift right] \end{tikzcd}$$

Interpreting symmetric residual

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$\odot$ preserves p.a+coh

Interpreting symmetric residual

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\begin{tikzcd}
\mathbb{C}^{\text{p.a + coh}} \arrow[rr, "\odot \text{ preserves p.a+coh}"'] \arrow[d, shift right=2] \arrow[d, phantom, "\sim"] & & x \odot \mathbb{C}^{\text{p.a + coh}} \arrow[d, shift right=2] \arrow[d, phantom, "\sim"] \\
(\mathrm{Pr}(\mathbb{C}), <, \#) \arrow[u, shift right=2] & & (\mathrm{Pr}(x \odot \mathbb{C}), <', \#') \arrow[u, shift right=2]
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Interpreting symmetric residual

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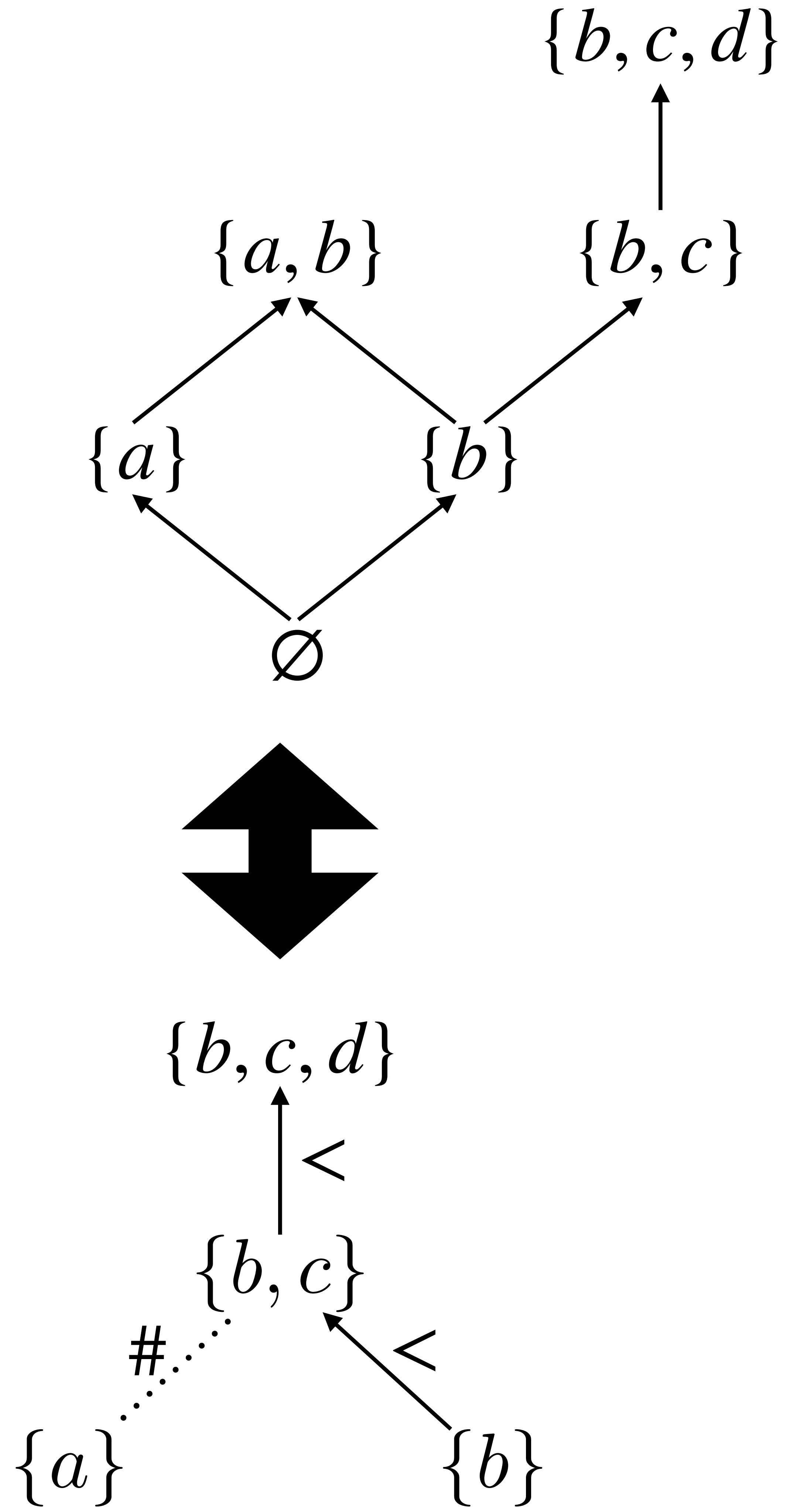
\mathbb{C}^{\text{p.a + coh}} \arrow[rr, "\odot \text{ preserves p.a+coh}"'] \arrow[d, shift right] & & x \odot \mathbb{C}^{\text{p.a + coh}} \arrow[d, shift right] \\
(\mathsf{Pr}(\mathbb{C}), <, \#) \arrow[u, shift right, "\sim"'] \arrow[rr, "\mathsf{Switch}_x"'] & & (\mathsf{Pr}(x \odot \mathbb{C}), <', \#') \arrow[u, shift right, "\sim"']
\end{tikzcd}
$$

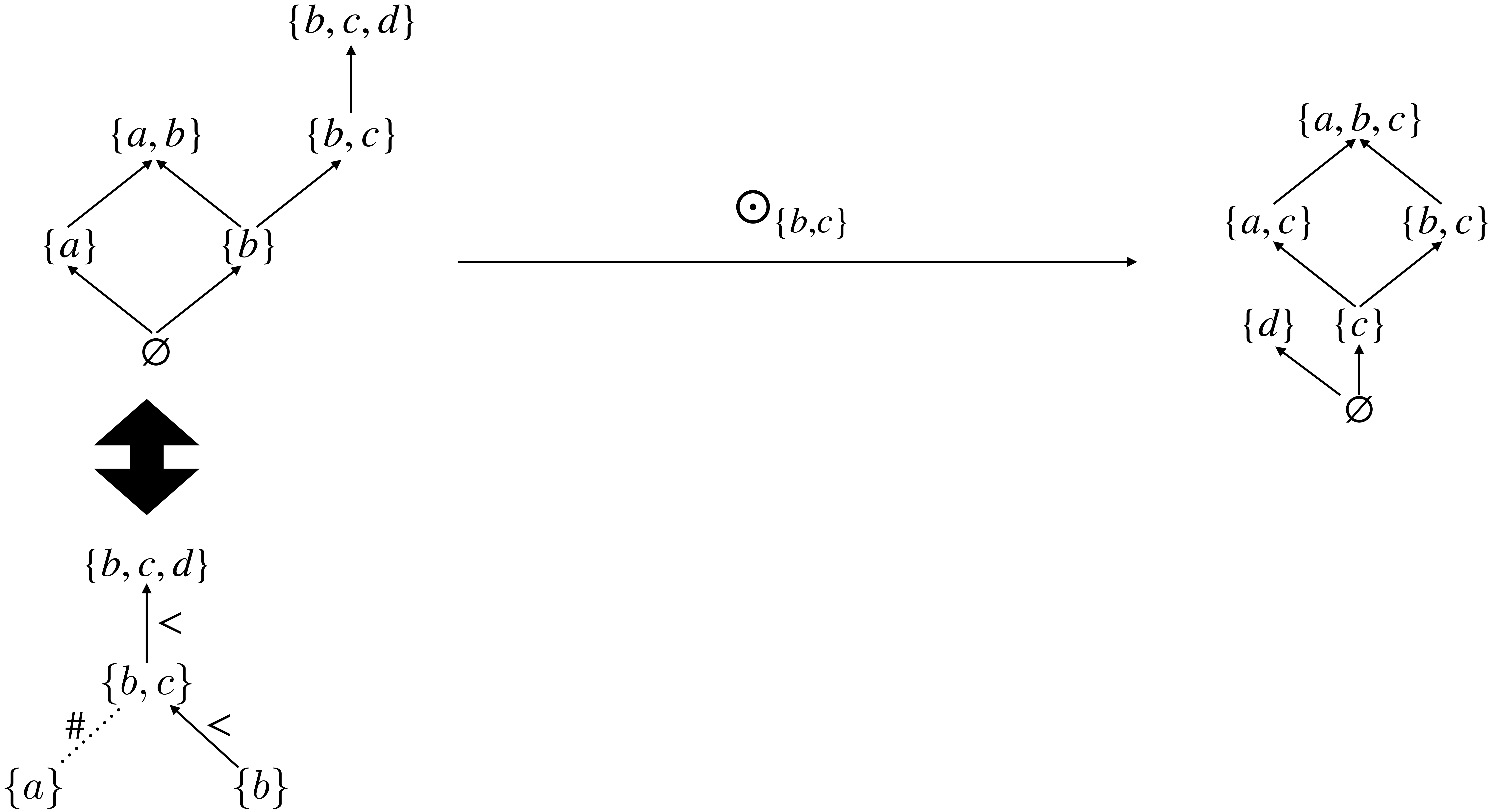
Event structure switch

$$\begin{tikzcd}
& & & \{b, c, d\} \\
& \{a, b\} & & \{b, c\} \arrow[u] \\
\{a\} \arrow[ur] & & \{b\} \arrow[ul] \arrow[ur] & \\
& \emptyset \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

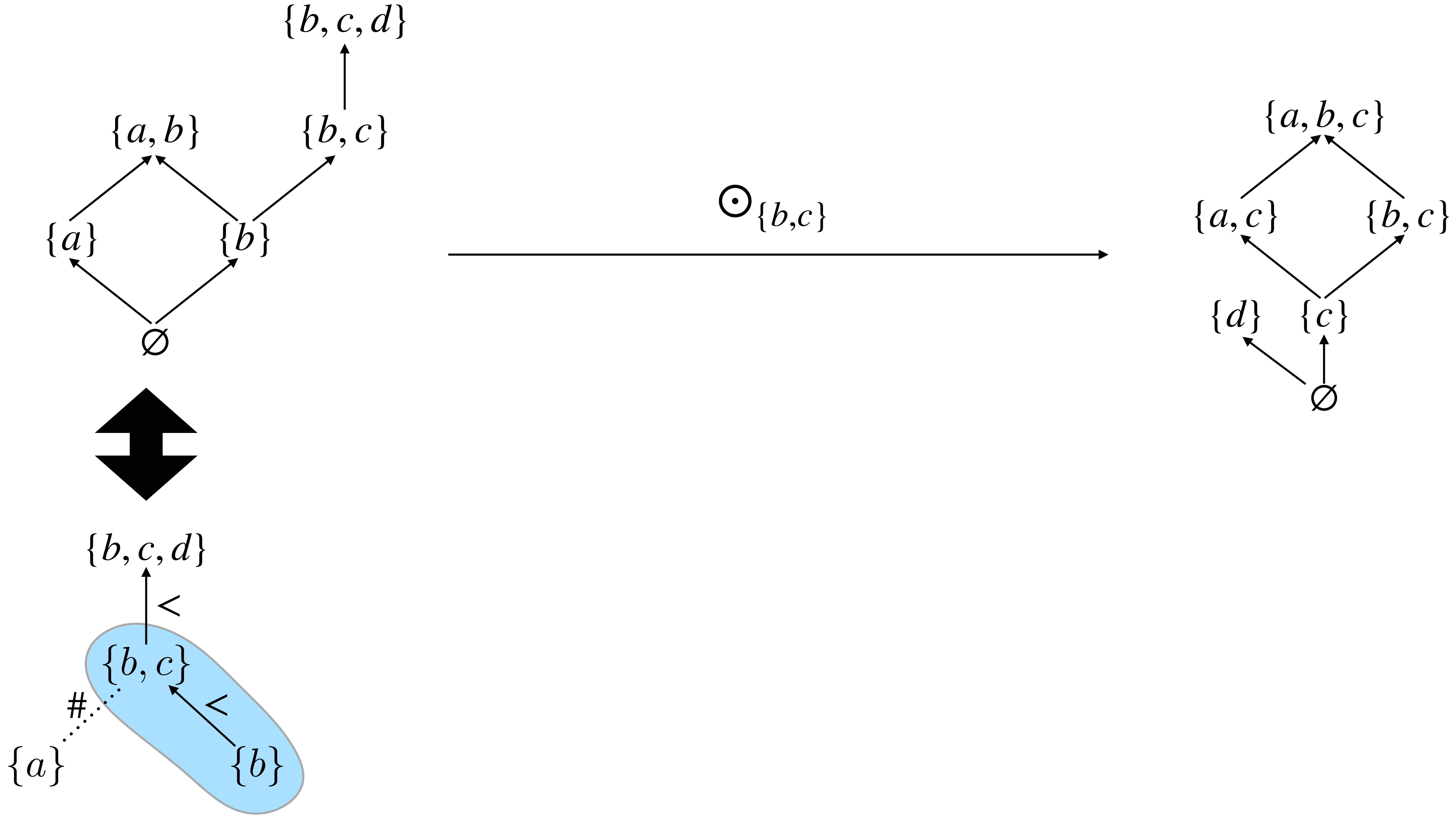
Event structure switch



Event structure switch



Event structure switch



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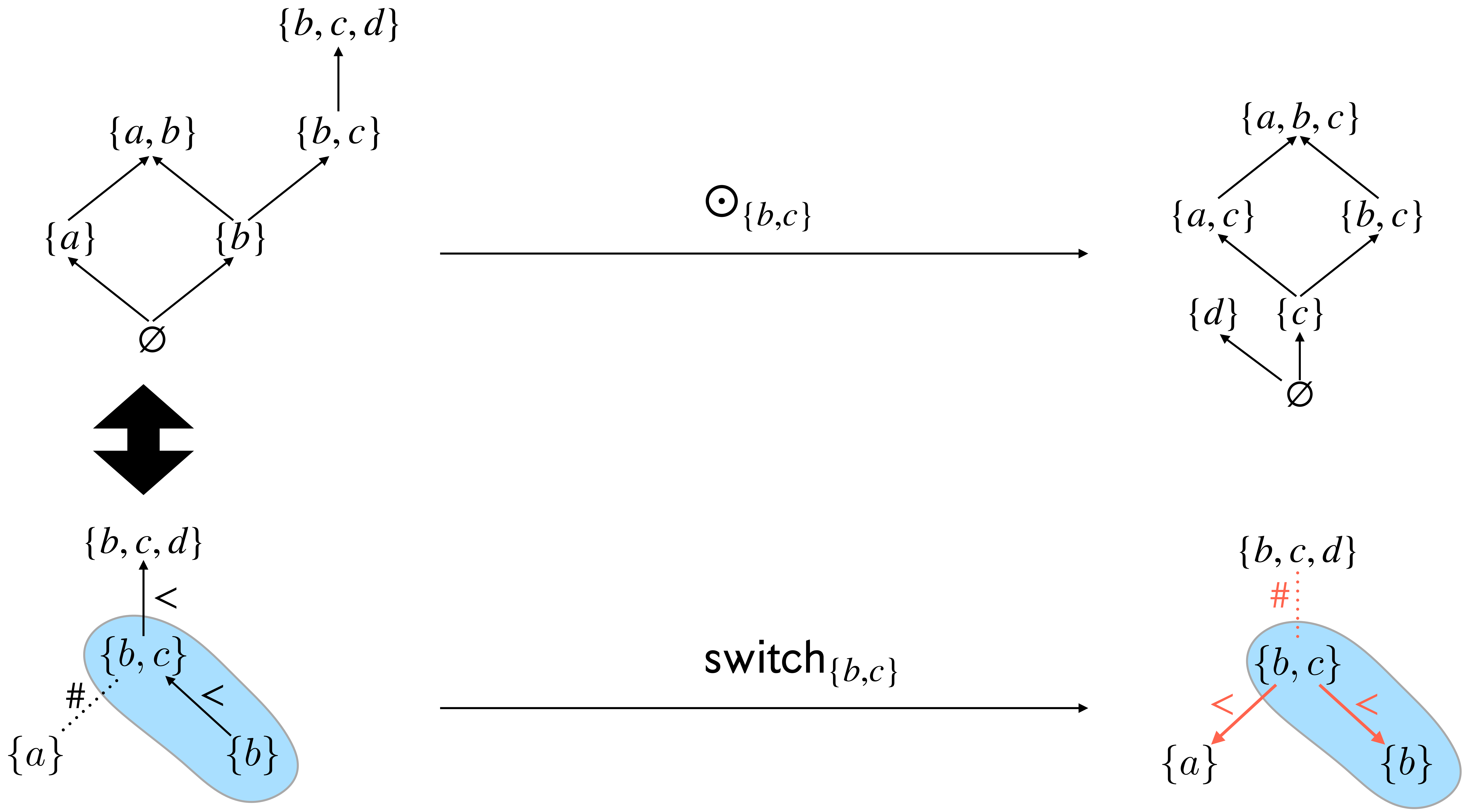
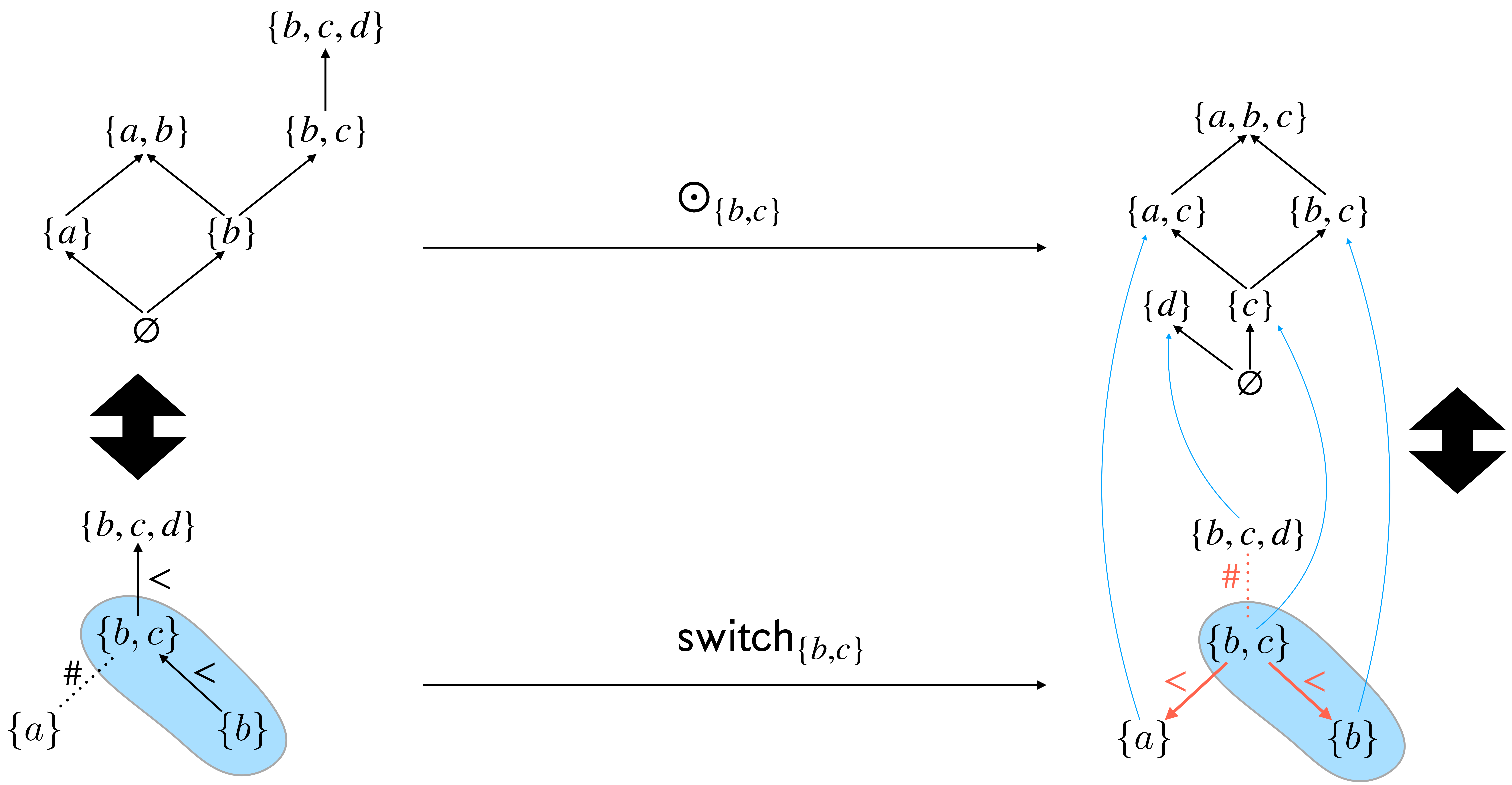


Event structure switch



Take away

Any concurrent formalism with stable (prime algebraic coherent) causal structure (CCS, CSP, Pi-calculus etc.) …

Can be equipped with a (causally consistent) reversible semantics by making sure transition steps implement a switch.

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Can be equipped with a (causally consistent) reversible semantics by making sure transition steps implement a switch.

$$\sum_i a_i \,.\, P_i \rightarrow_{a_j} P_j + a_j \,.\, \sum_{i \neq j} a_i \,.\, P_j \quad \text{(rsum)}$$