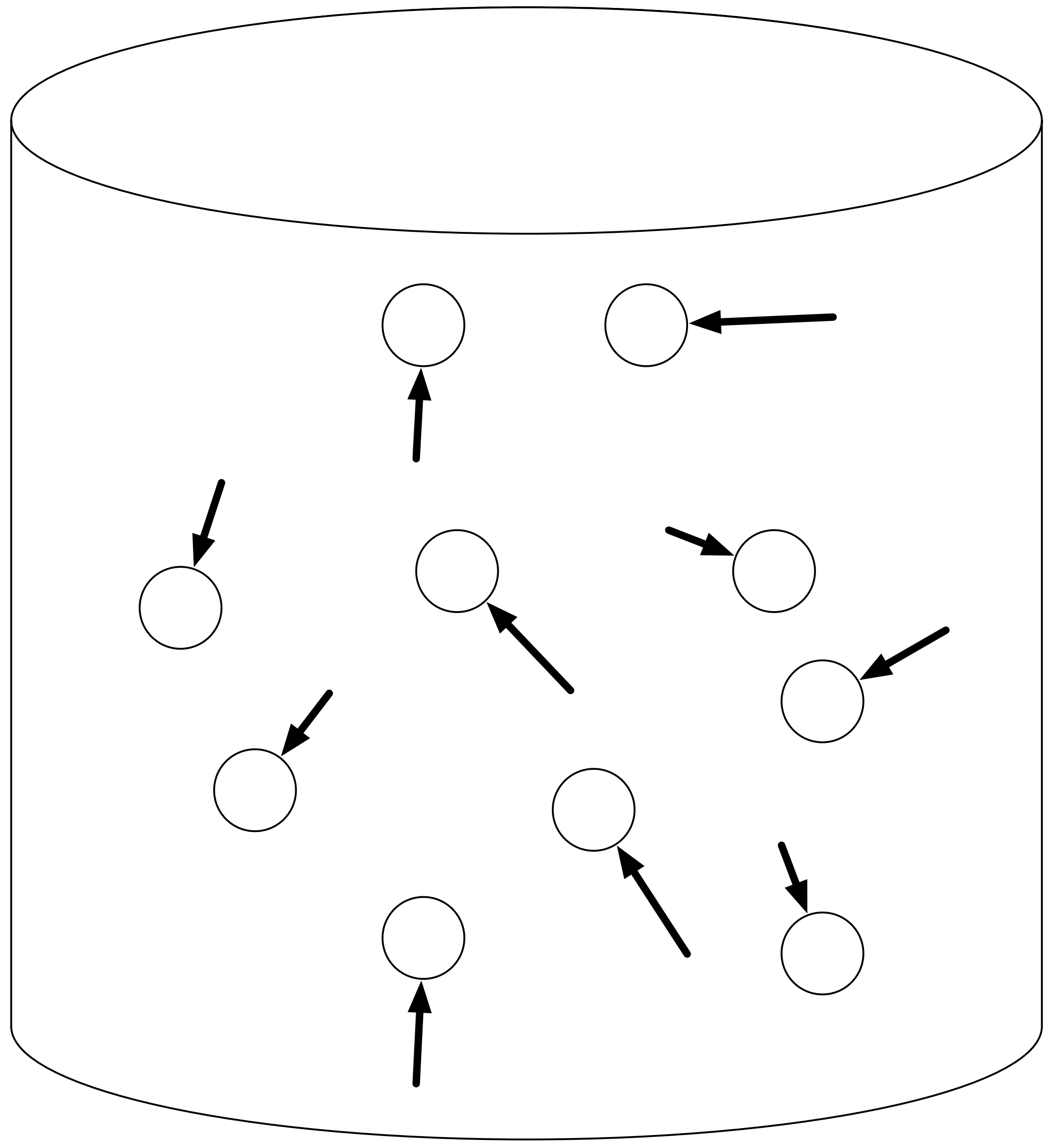
Reversible computations are computations

Clément Aubert (Augusta Univ, USA), **Jean Krivine** (CNRS, IRIF Paris)

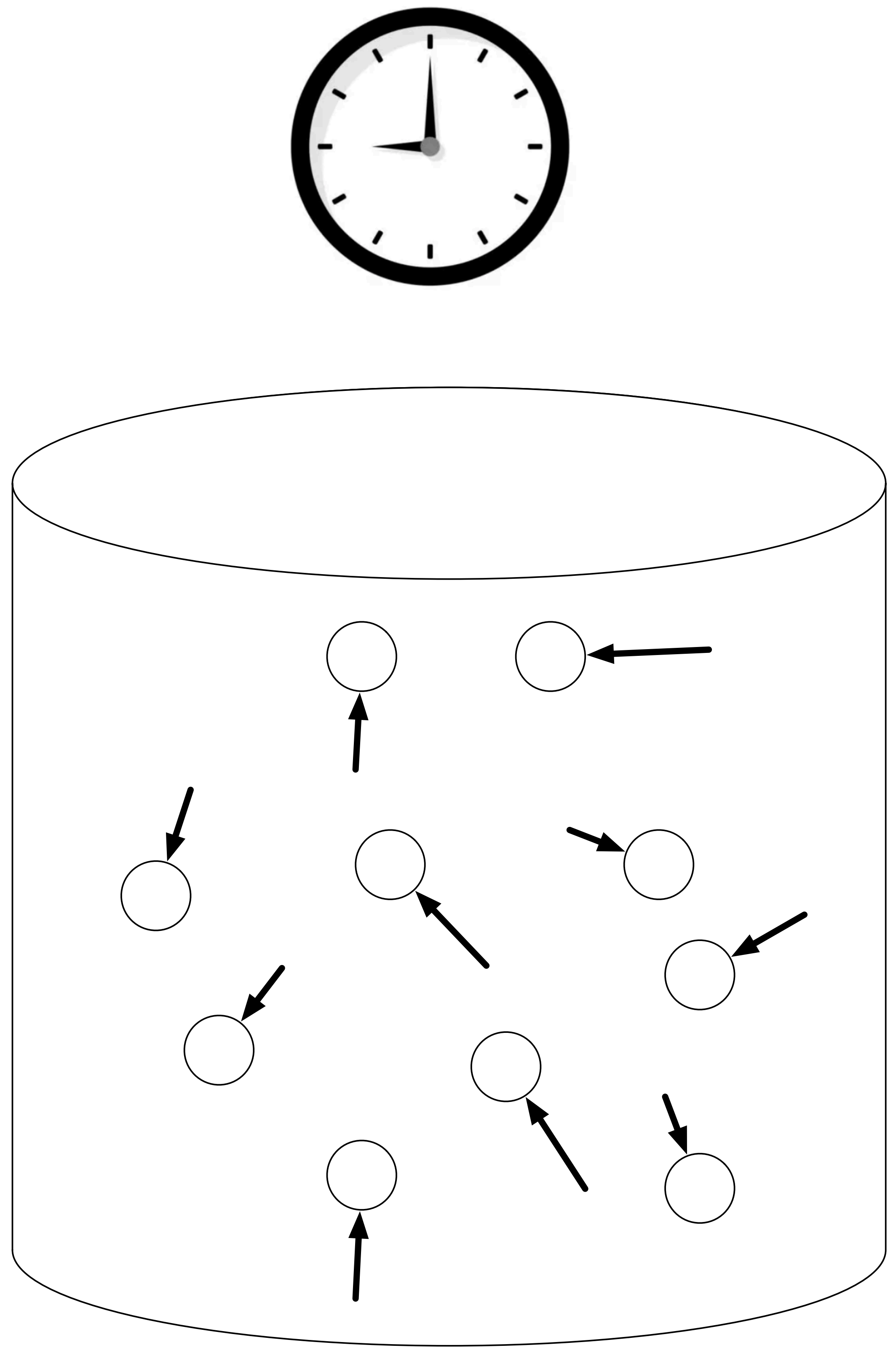
PPLV Seminar

Dynamical Systems



Move and collide
(local interactions)

Dynamical Systems



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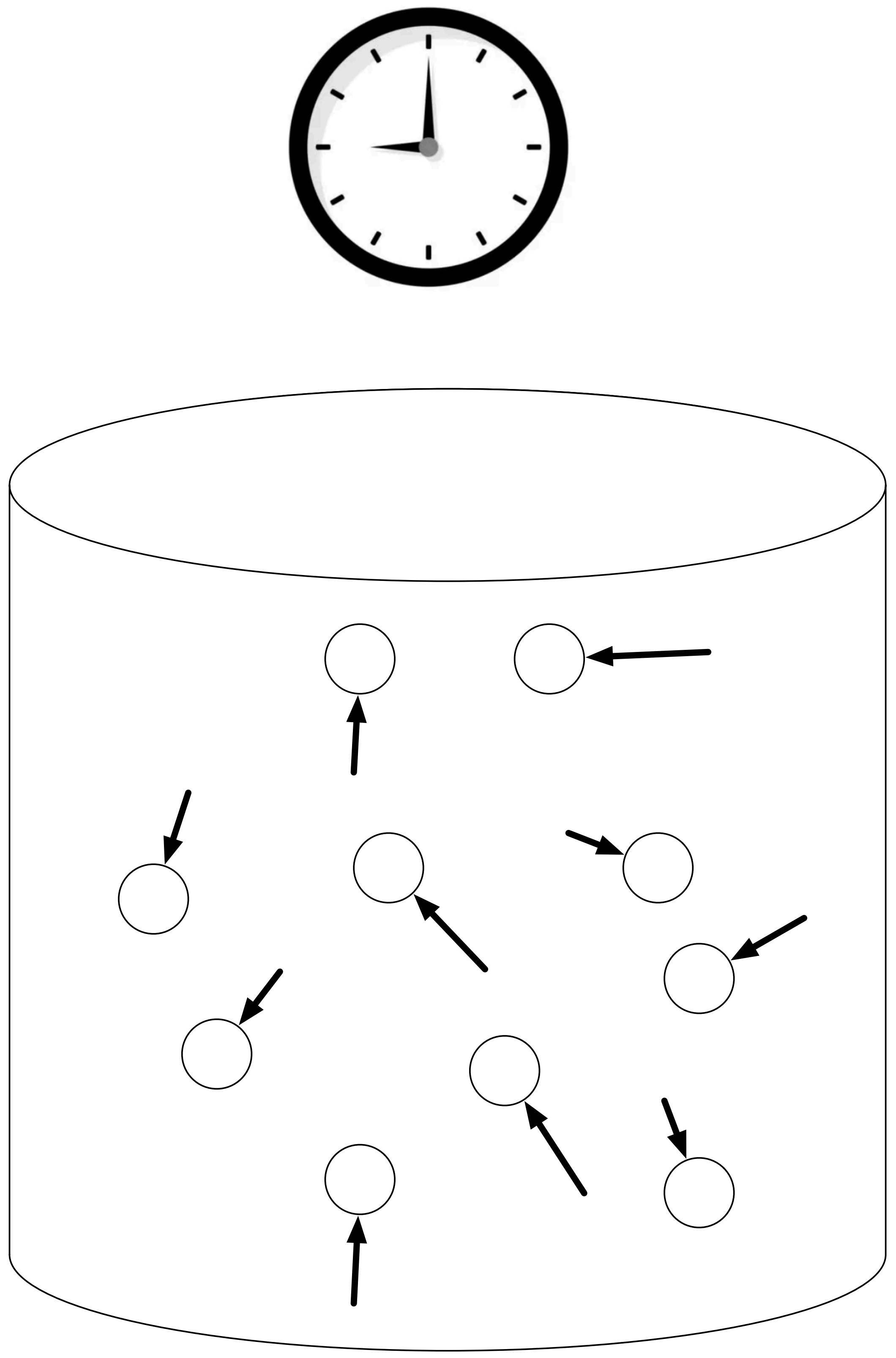

Move and collide
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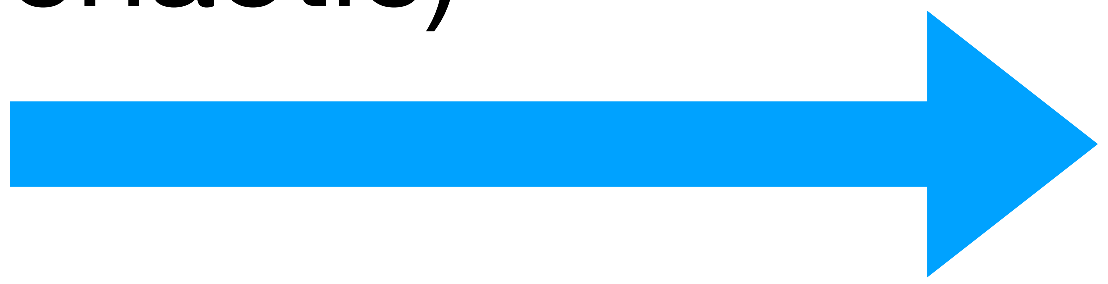
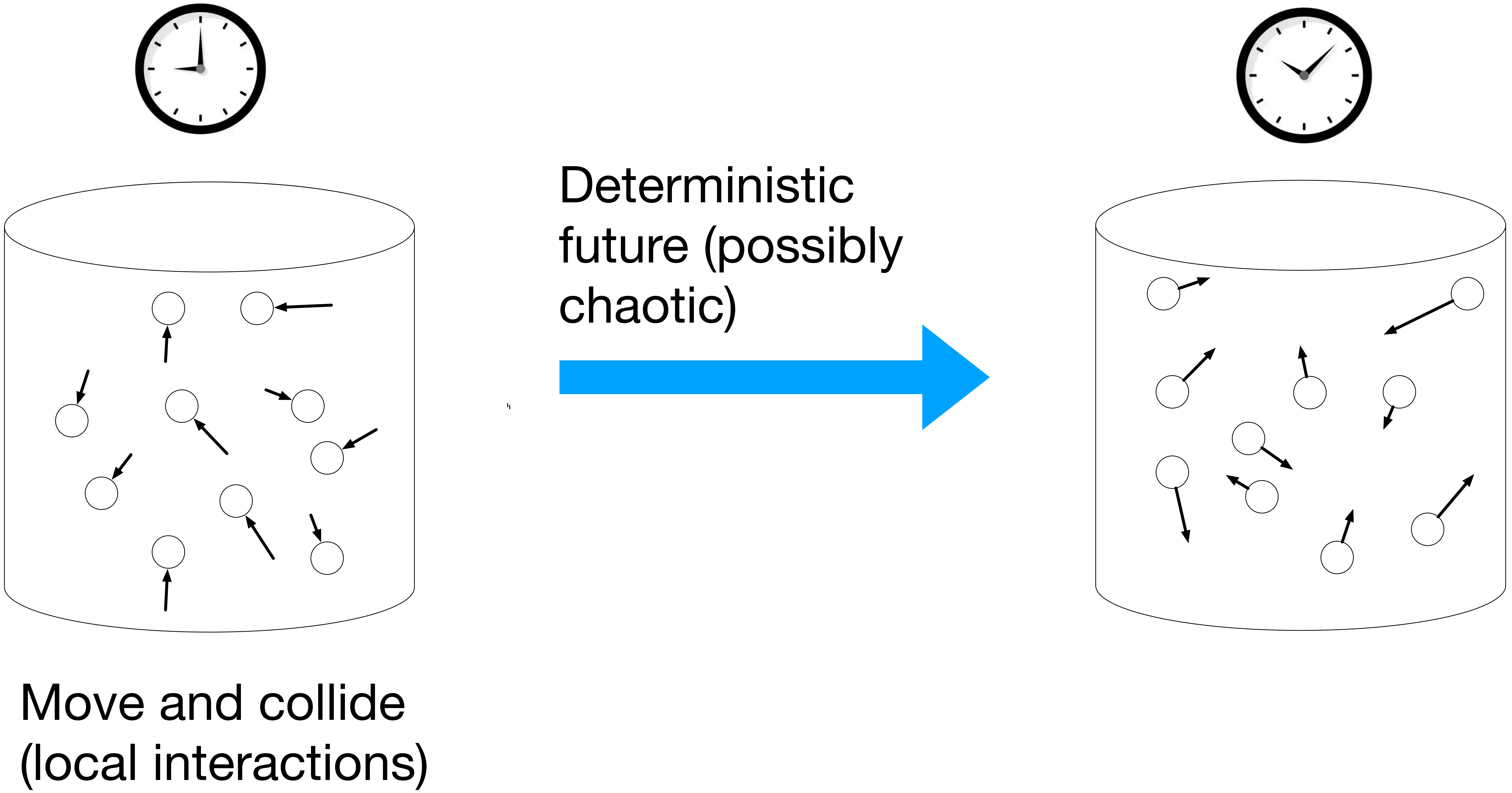
Deterministic
future (possibly
chaotic)



Dynamical Systems



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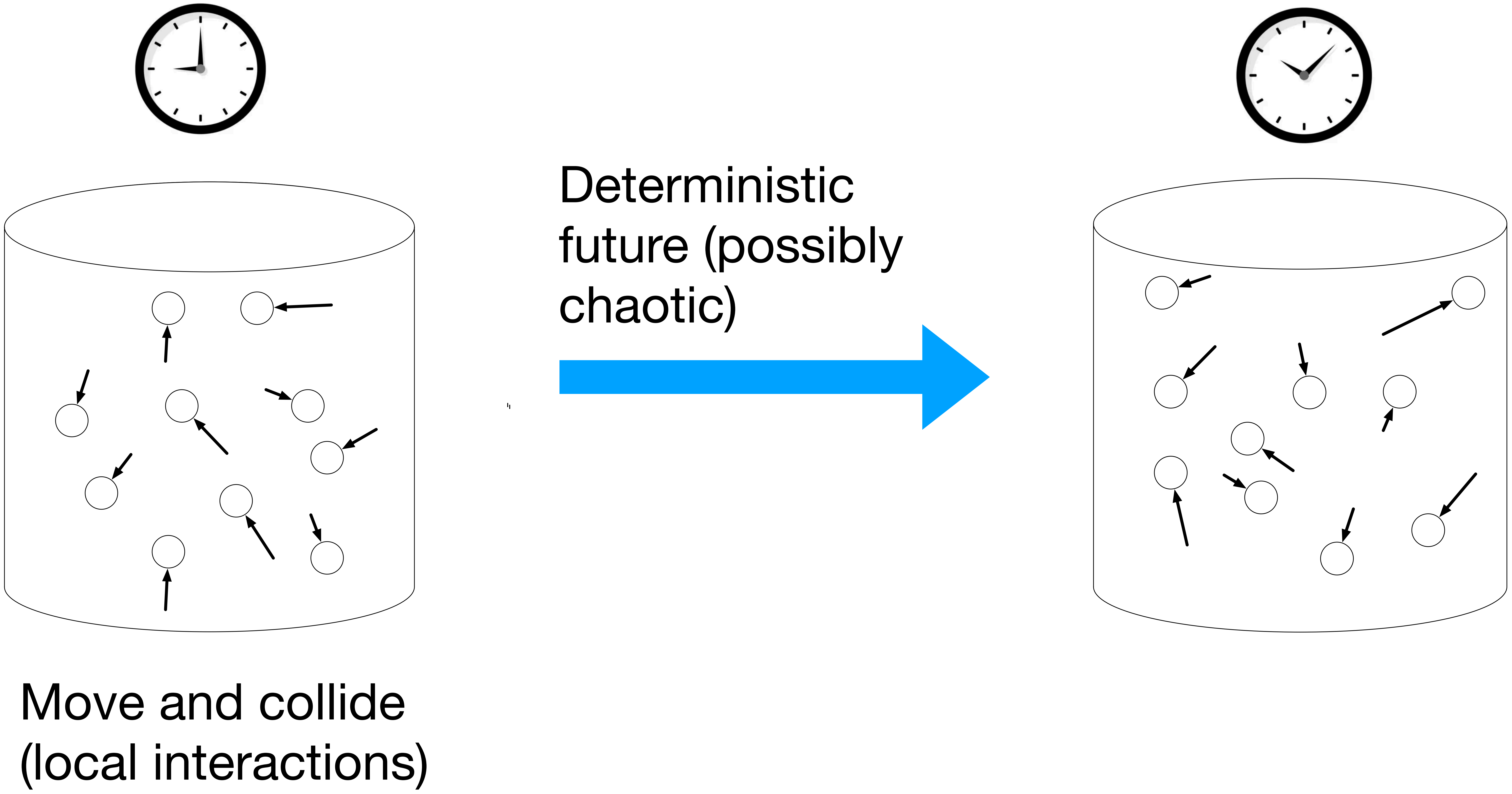

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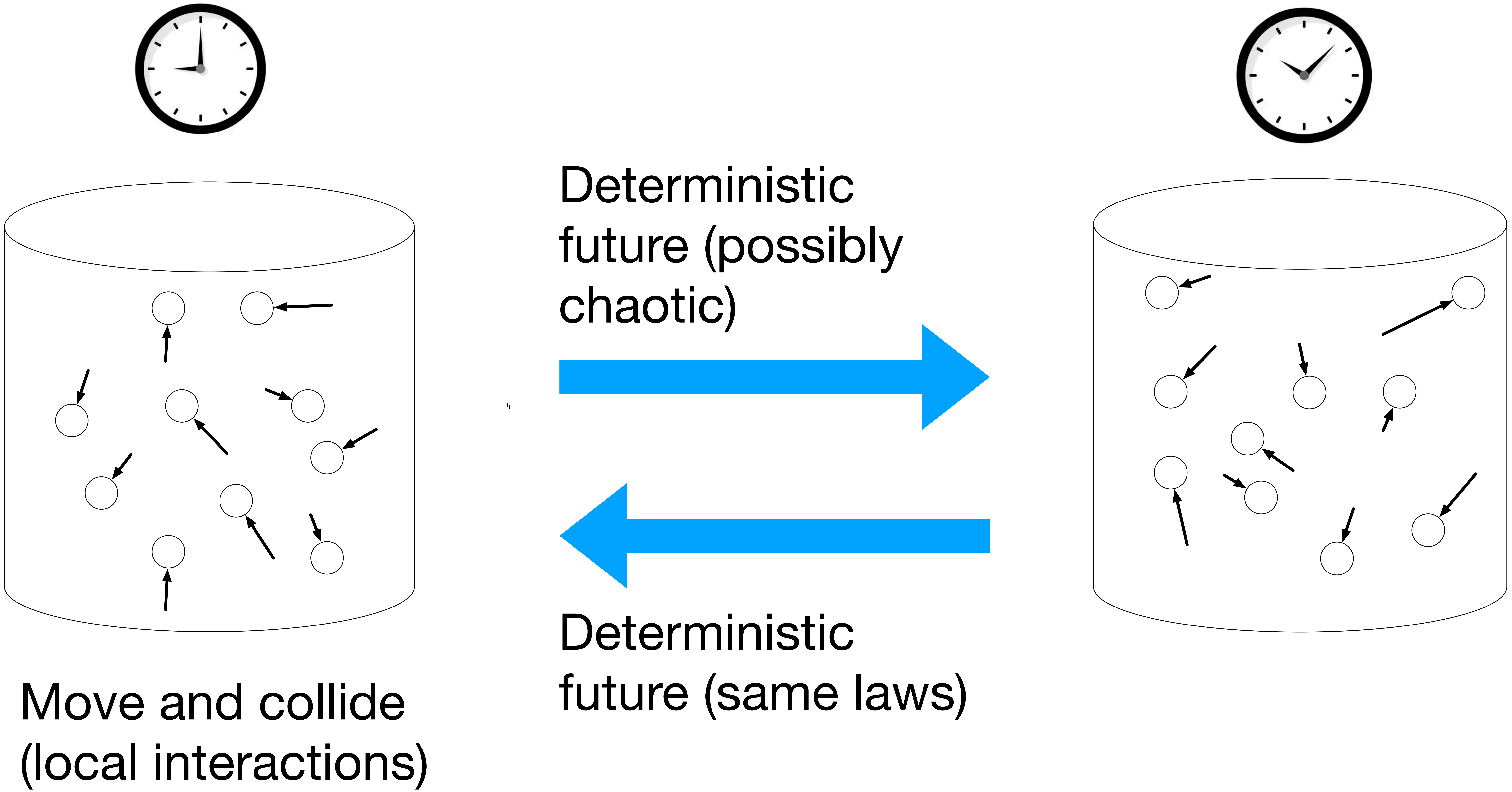
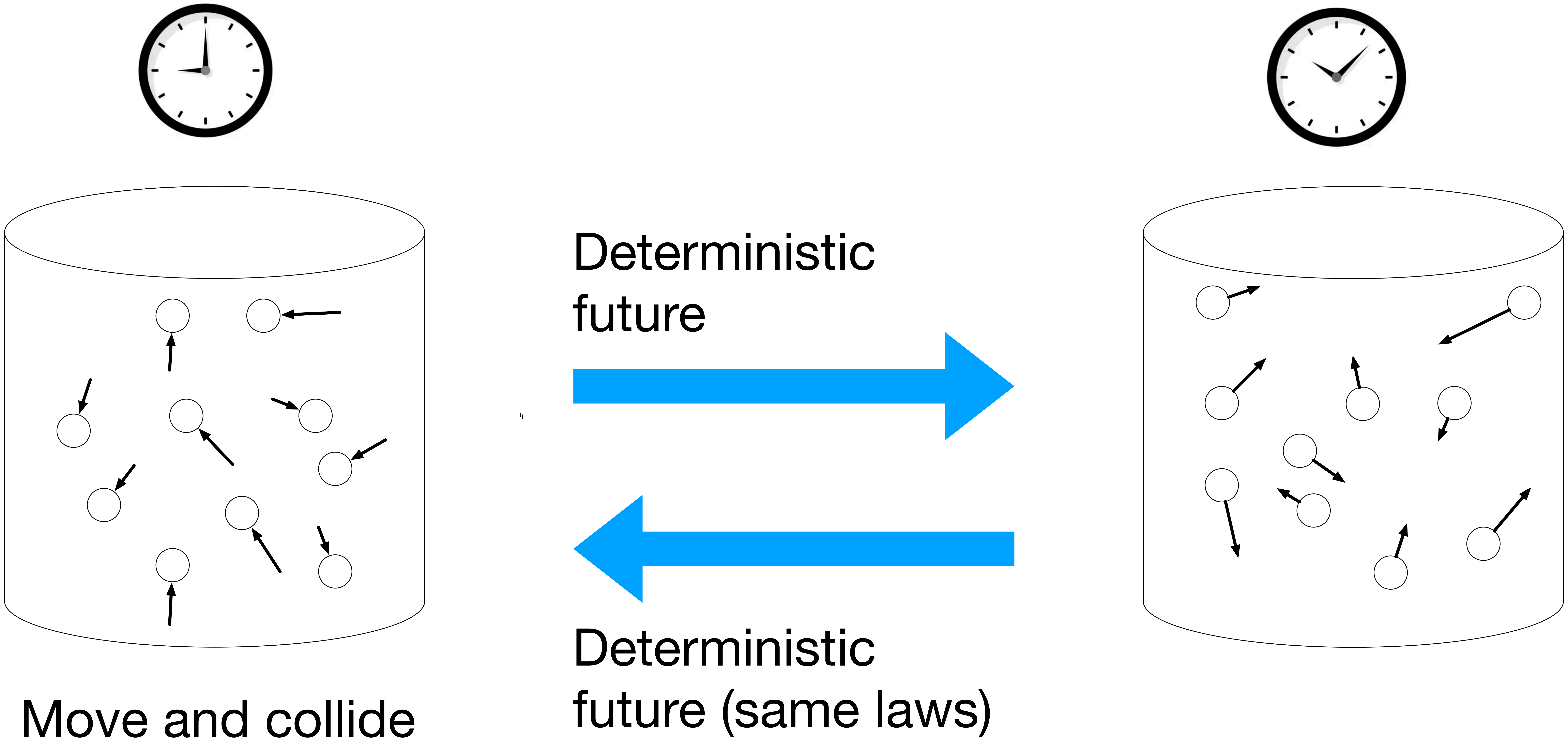


Concurrent Systems



Move and collide
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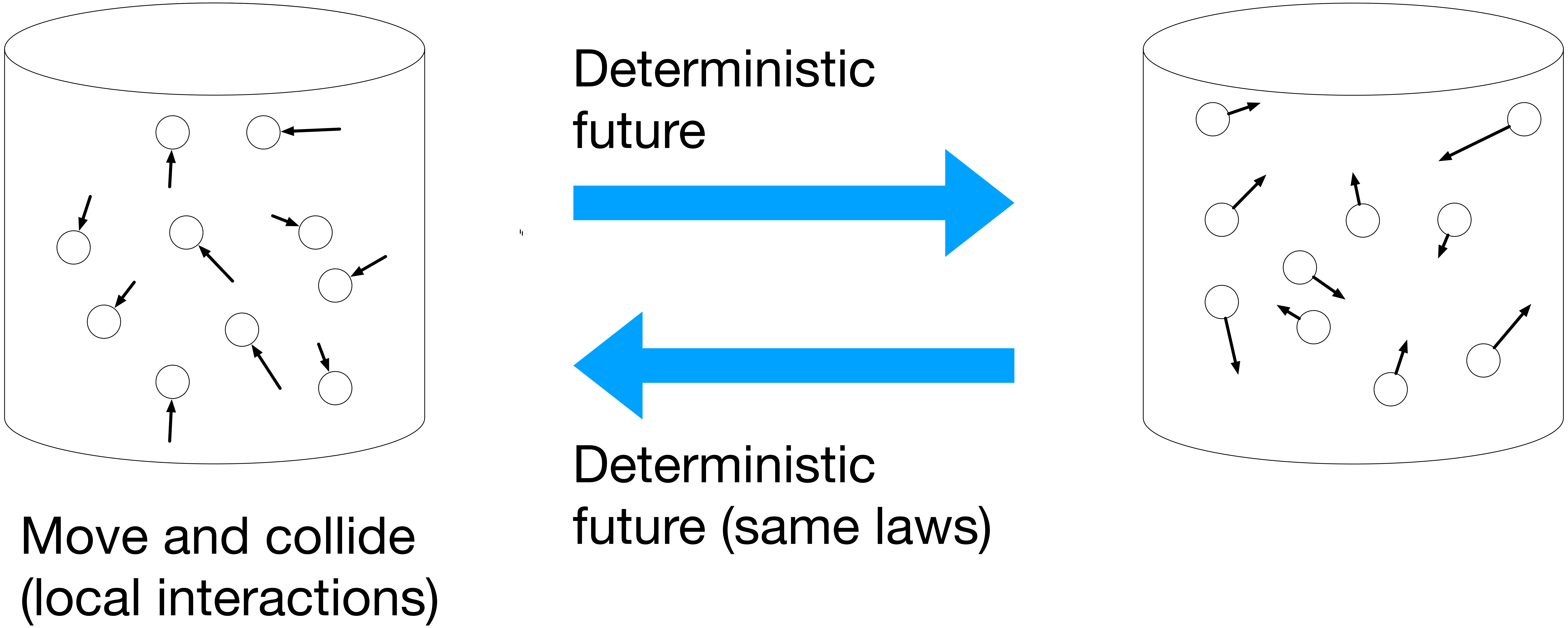
Concurrent Systems

Quantify over all possible schedulings



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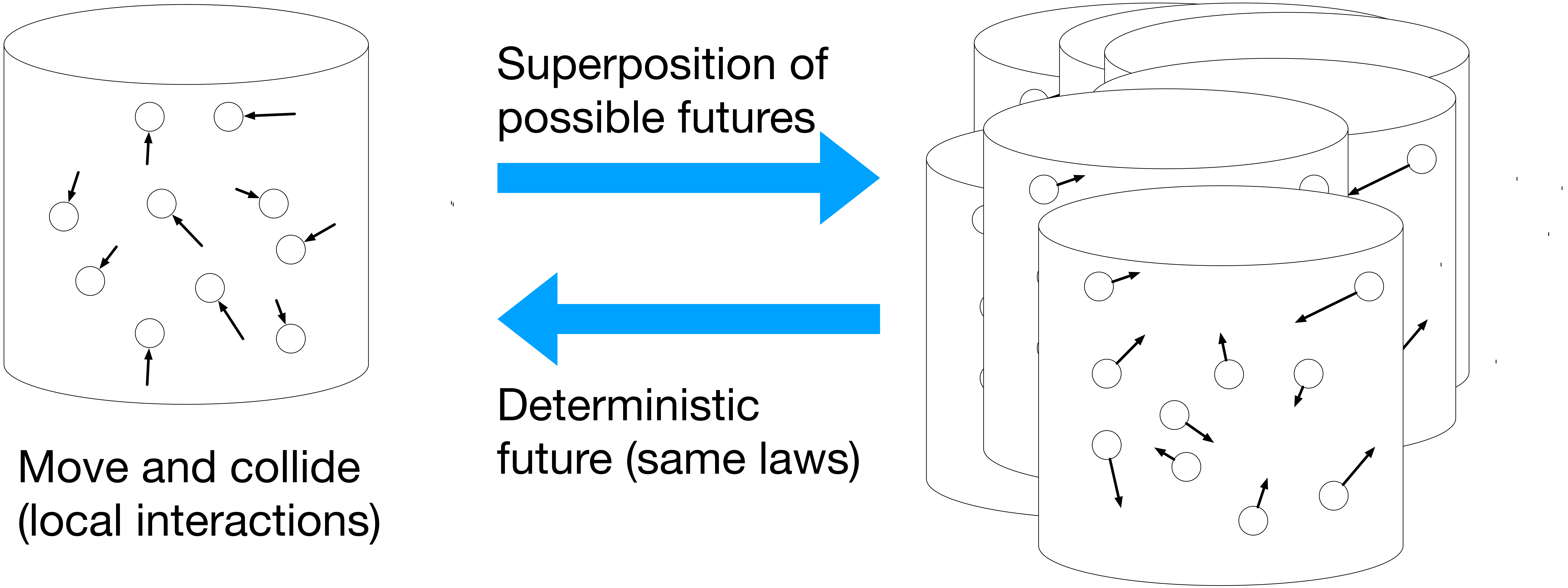

Concurrent Systems

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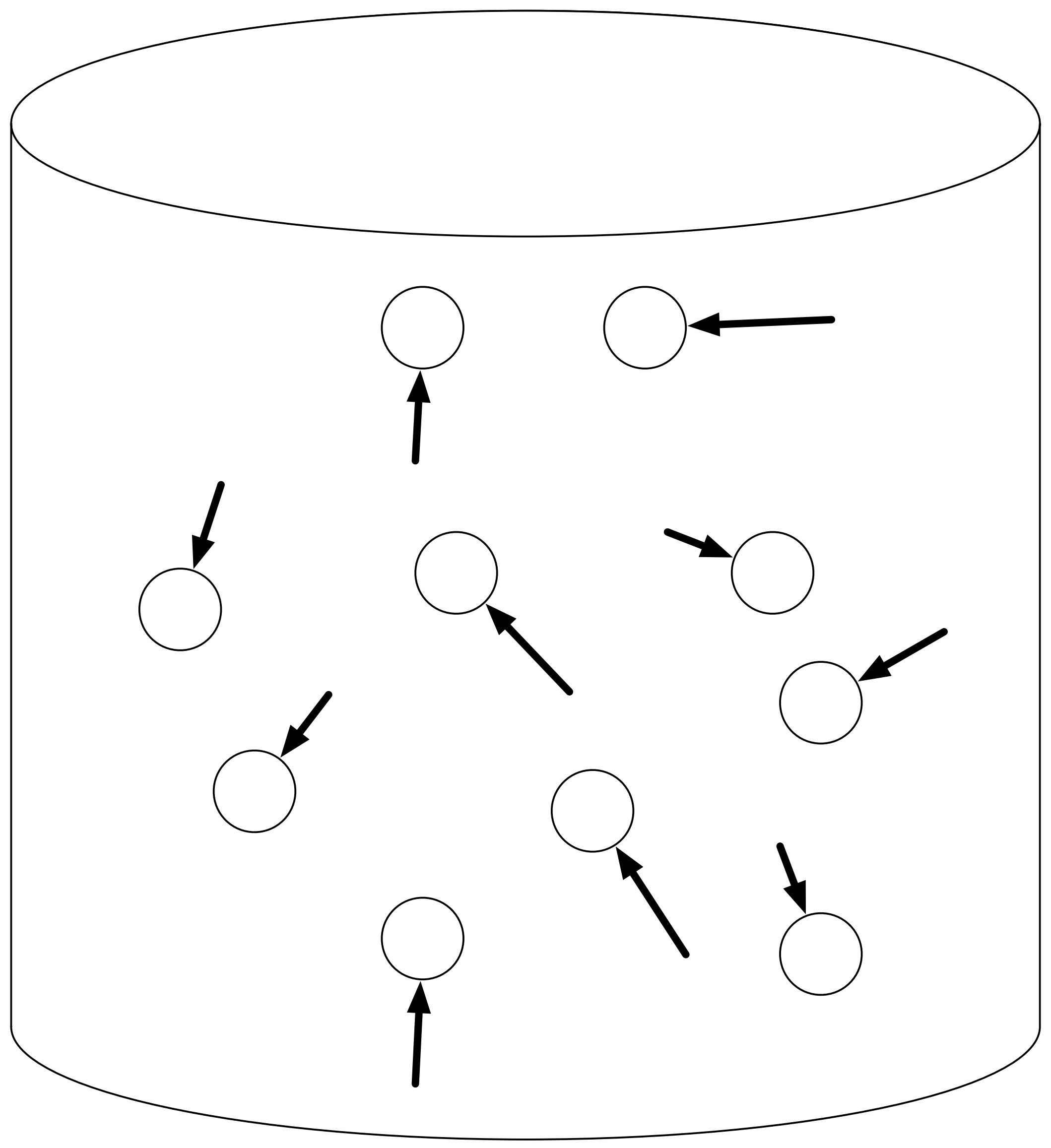
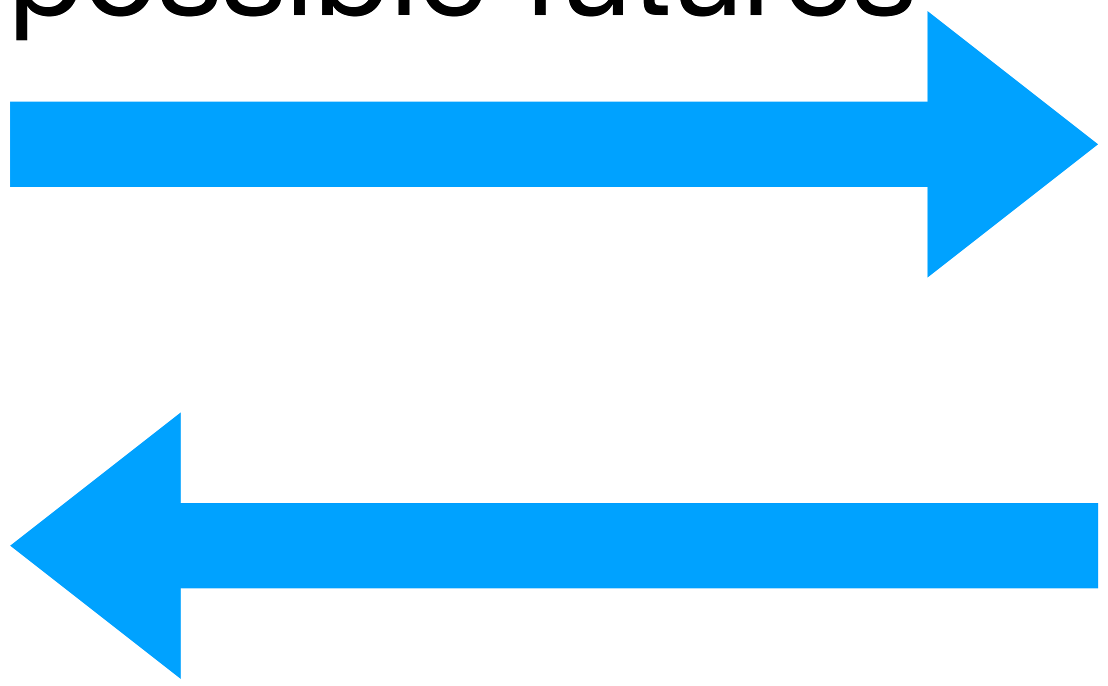
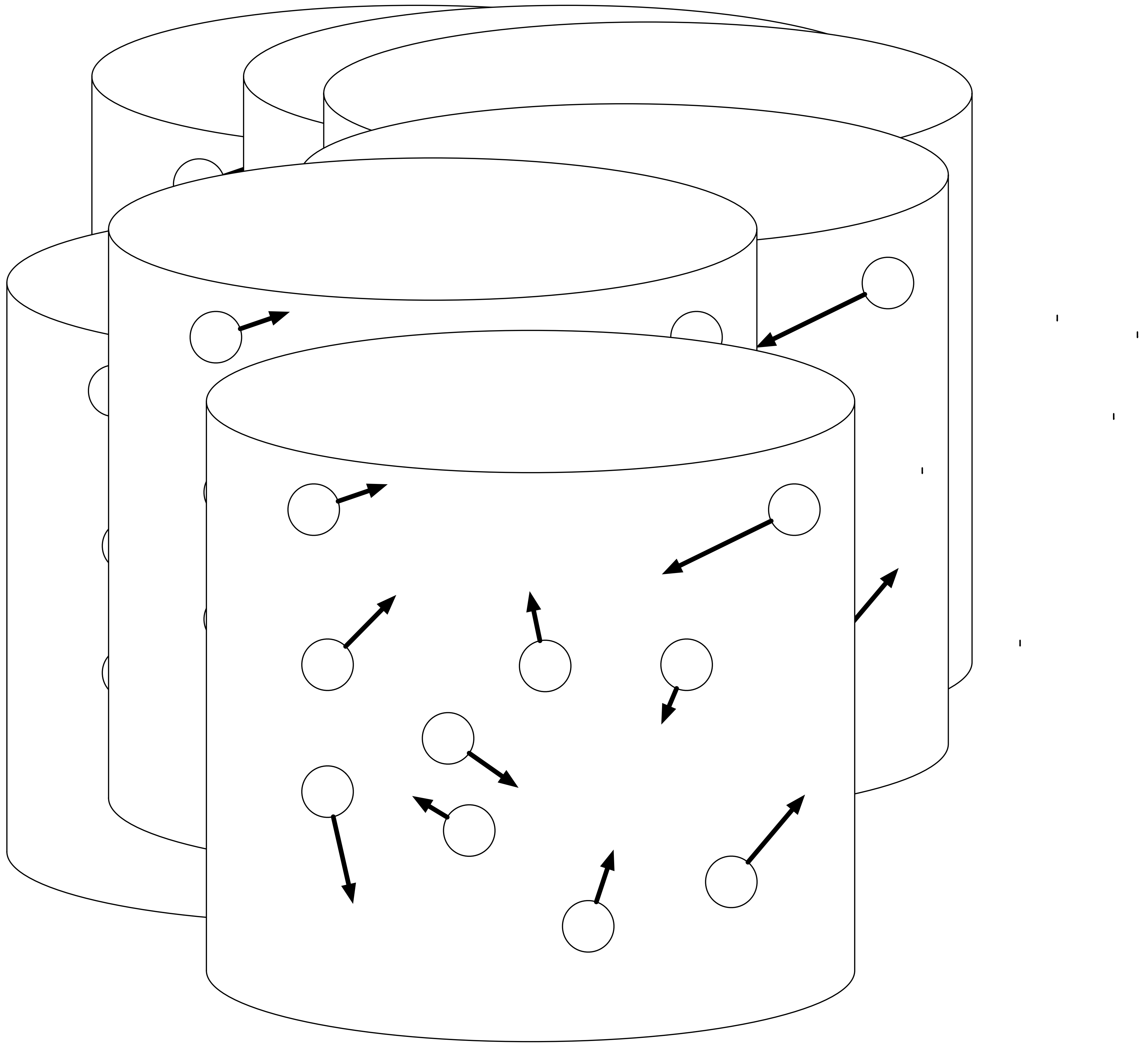


Move and collide
(local interactions)

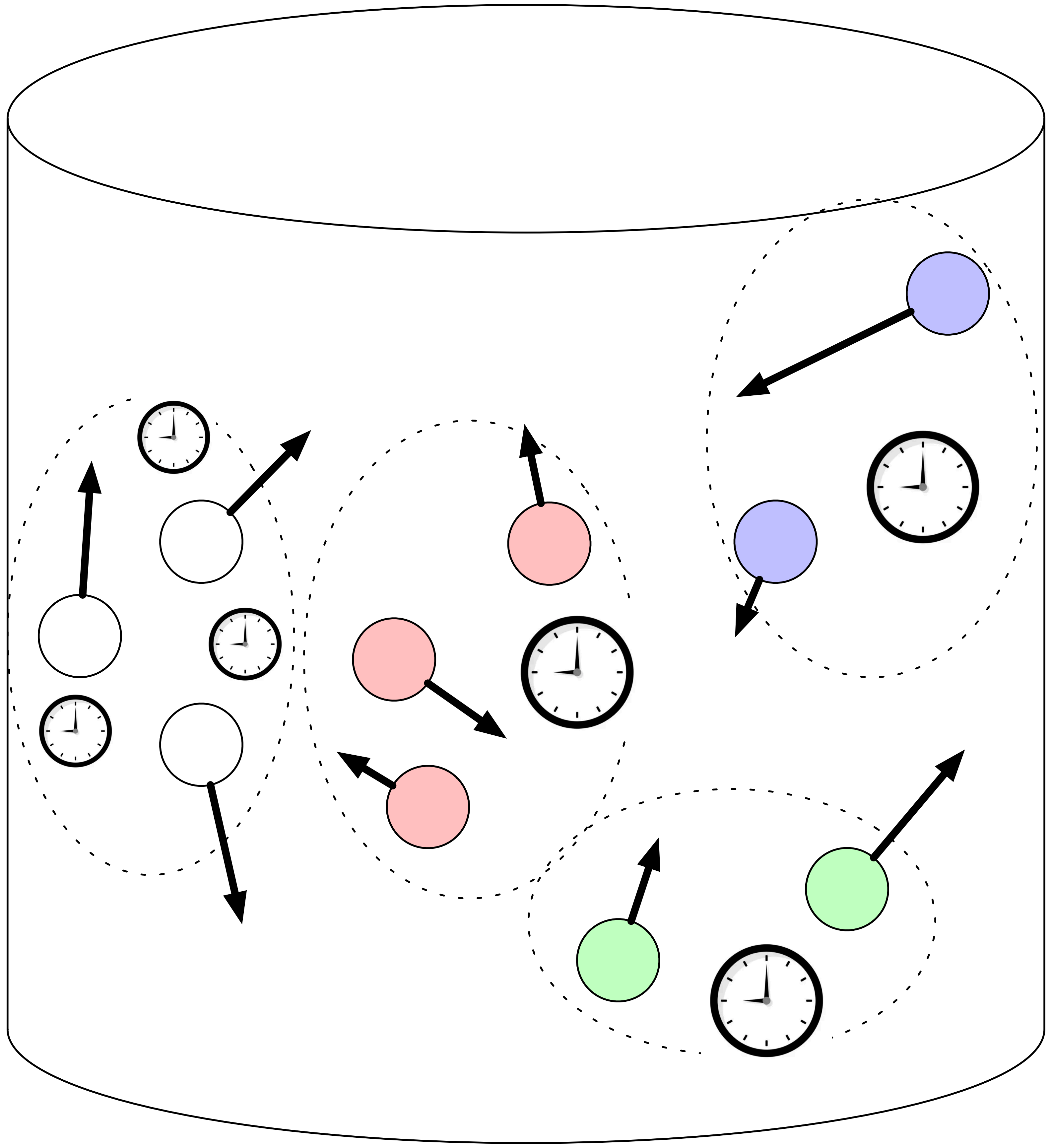
Superposition of
possible futures



?



Causal consistent reversibility

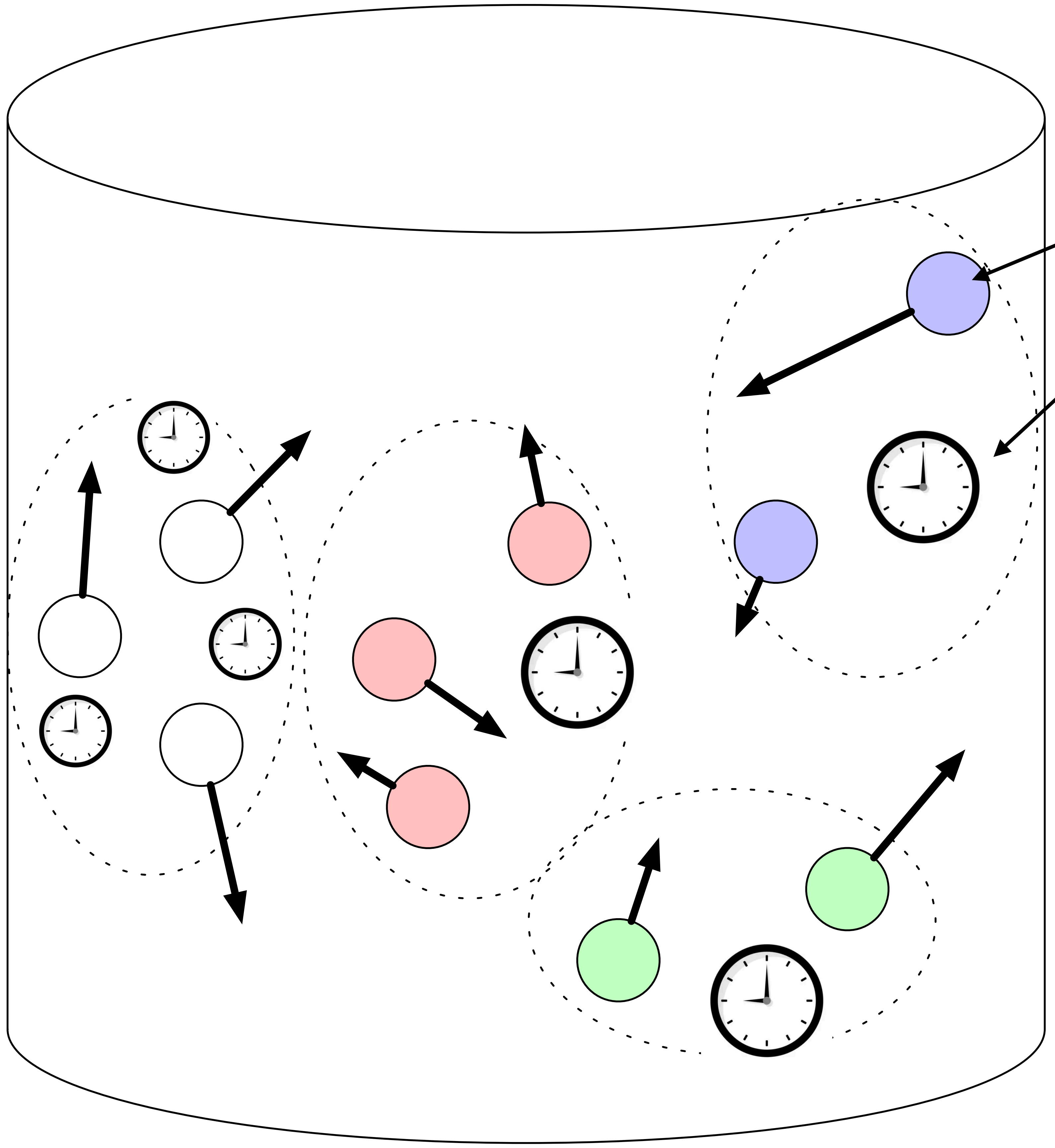


A subsystem may be “backwards”, while the rest is “forwards”…

But agents can interact only if they are in compatible space-time

Lamport’s clocks (78): causality as abstract time

Causal consistent reversibility



Agent cannot backtrack beyond this time point alone to avoid a paradox…

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But agents can interact only if they are in compatible space-time

Lamport's clocks (78): causality as abstract time

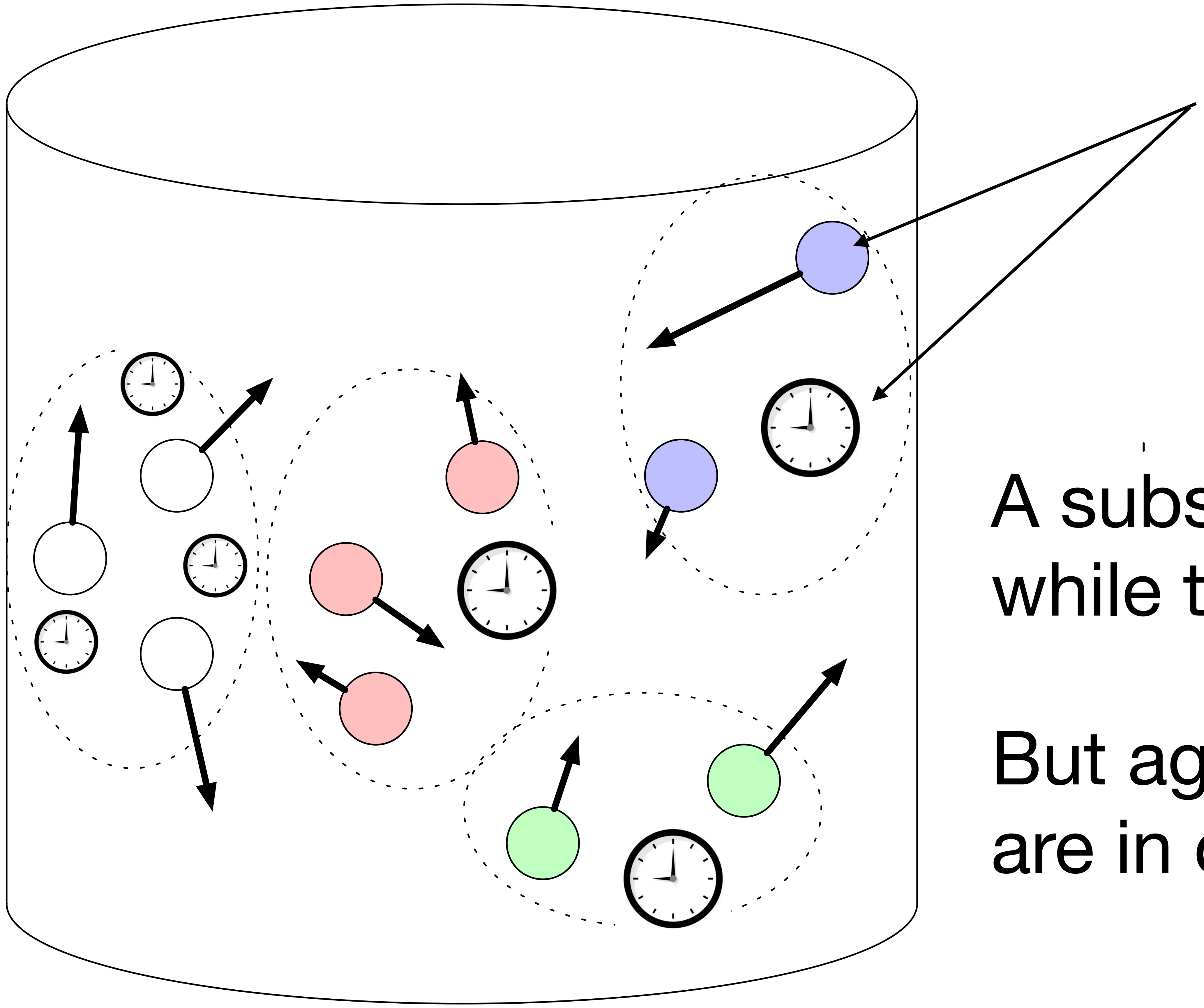
Causal consistent reversibility

Transactional systems

Concurrent debuggers

Counterfactual reasoning

Self assembly



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But agents can interact only if they are in compatible space-time

Lamport’s clocks (78): causality as abstract time

Well understood operationally

Pi $\bar{a}b.P \mid a(x).Q \to P \mid Q\{b/x\}$

Computation is dissipative

Well understood operationally

Pi $\bar{a}b.P \mid a(x).Q \rightarrow P \mid Q\{b/x\}$

Computation is dissipative

explicit subst.

RPi $m : \bar{a}b.P \mid m' : c(x).Q \leftrightarrow (i, \bar{a}b).m : P \mid (i, c[d/x]).m' : Q$

Computation is information preserving!

with
$$\begin{aligned} m'(c) &= m(a) \\ m(b) &= d \end{aligned}$$

subst. in m applied to b

Well understood operationally

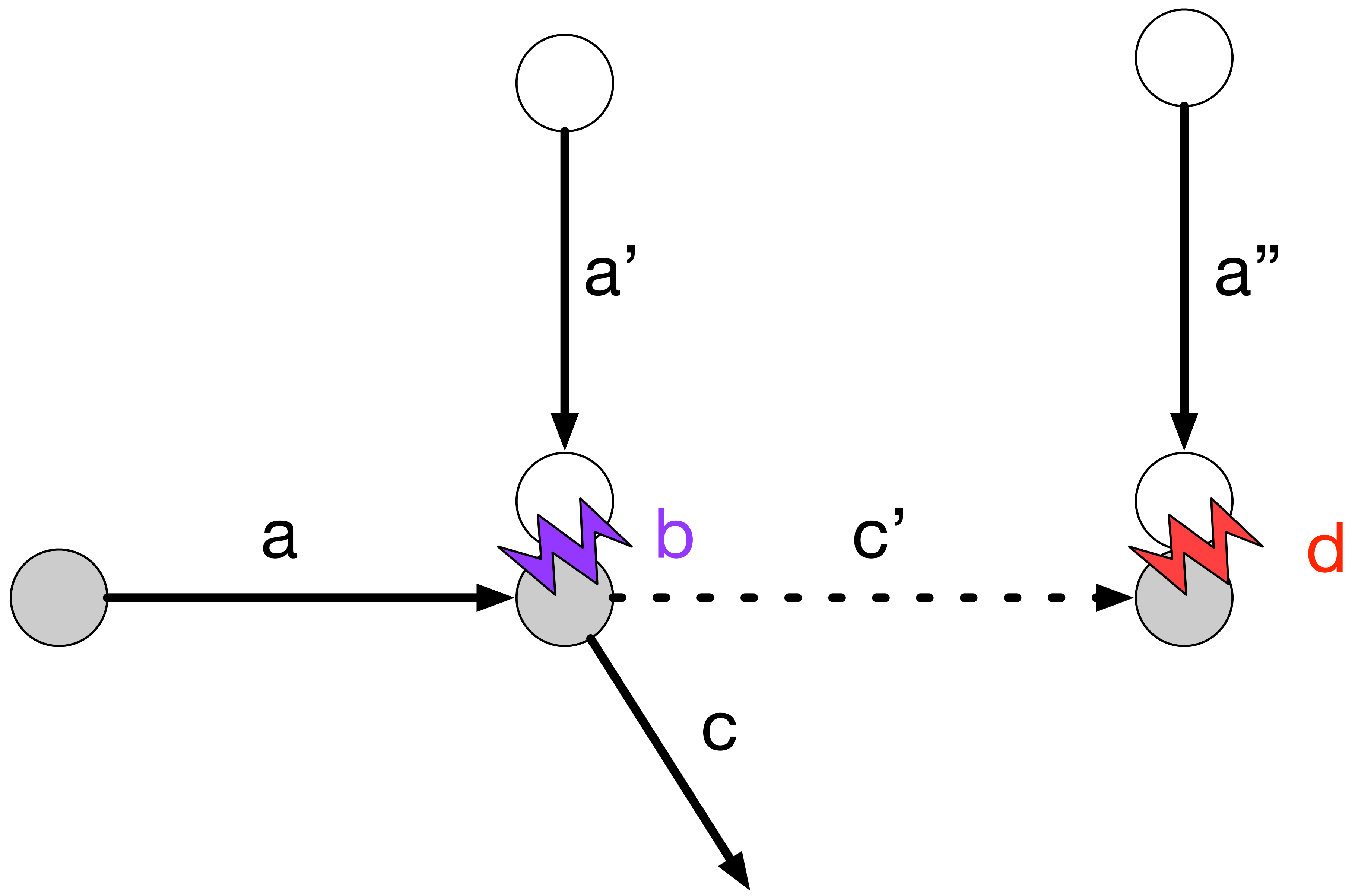
XOR

x	y	z
0	0	0
0	1	1
1	0	1
1	1	0

CNOT

x	y	z	m
0	0	0	0
0	1	1	1
1	0	1	0
1	1	0	1

Aim: a denotational view of reversibility



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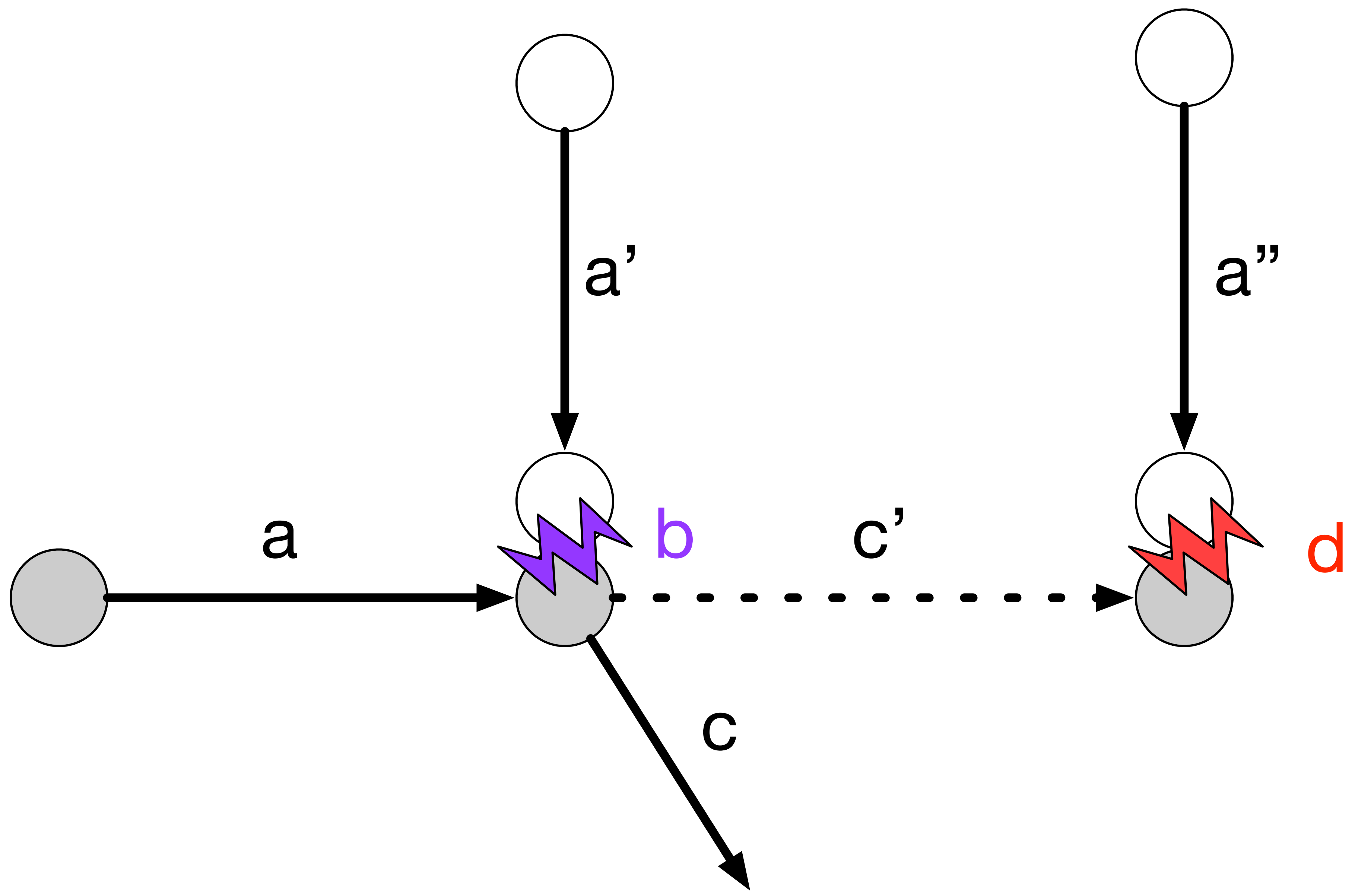

Logical characterisation

Time	$\{a, a'\} < b \ \wedge b < c$
$\varnothing$	$a' \ \# \ c' \quad a < c'$
	$\{a'', c'\} < d$

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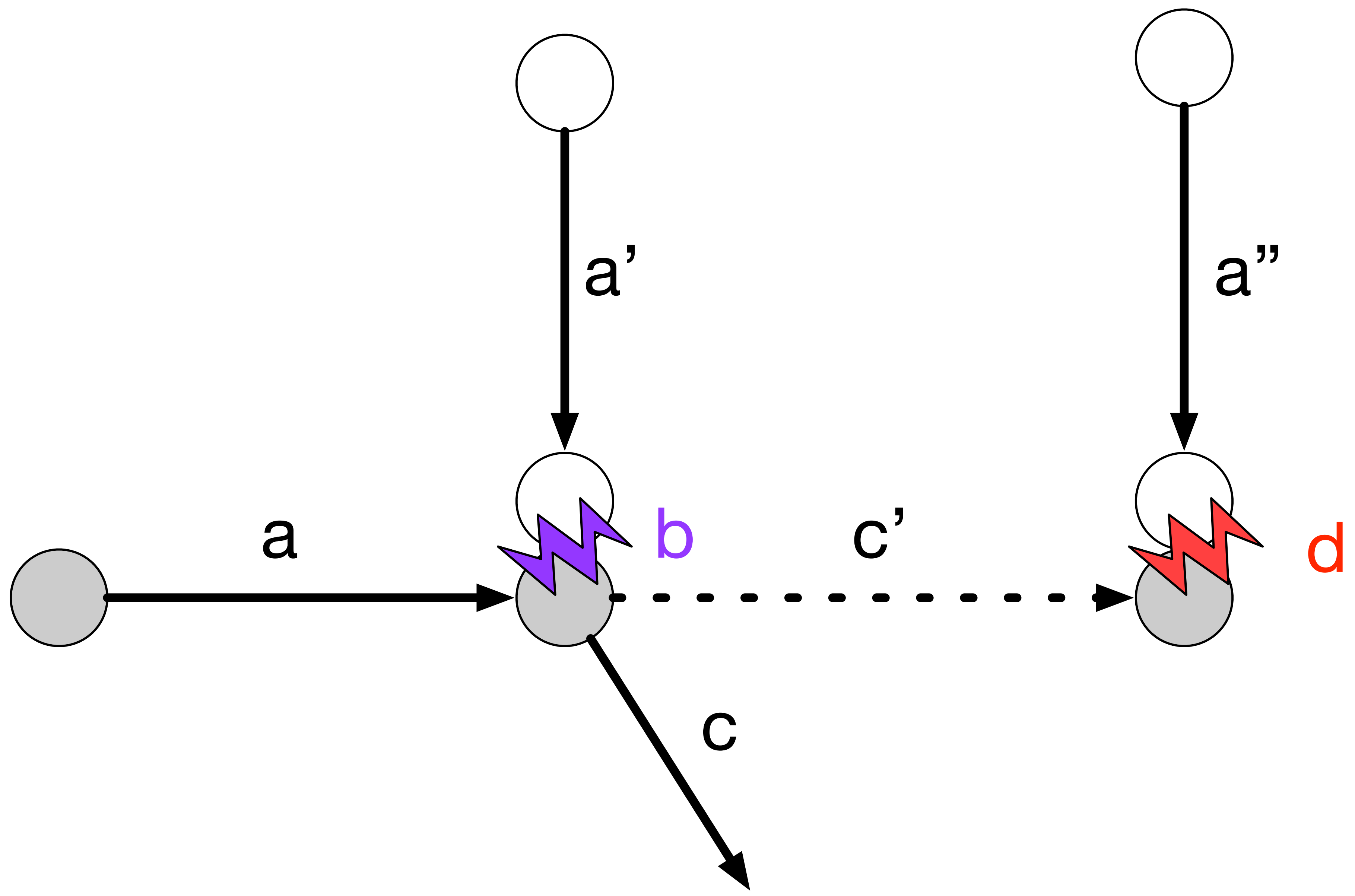

Logical characterisation

Time

$\varnothing$

$\{a, a'\} < b \ \wedge b < c$

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Time

$\{a, c', a'', d\}$

$\varnothing$

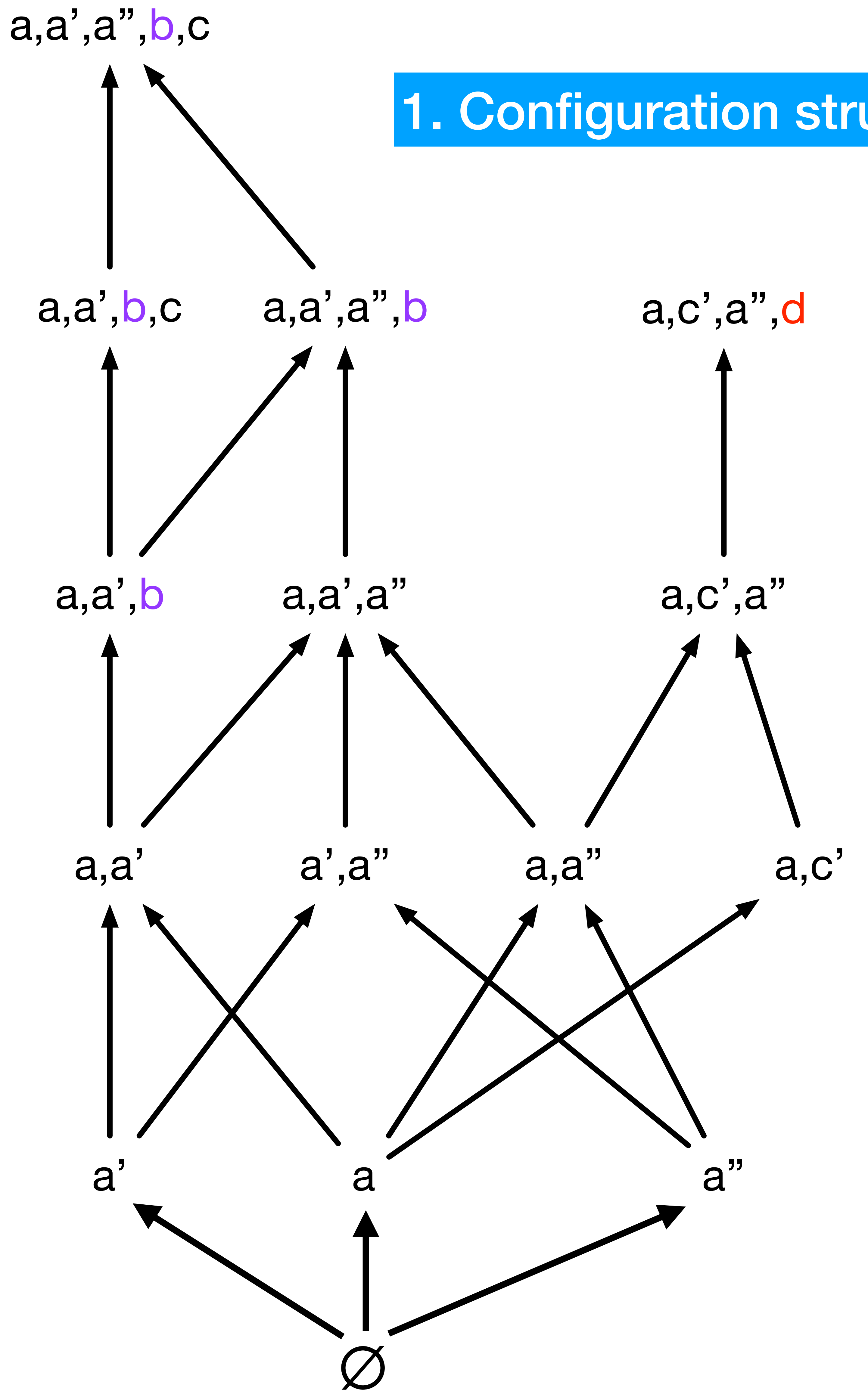
$d_* < \{a''_*, c'_*\}$

$c'_* < a' \quad c'_* < a_*$

$a' < b \wedge b < c$

Aim: a denotational view of reversibility

1. Configuration structures



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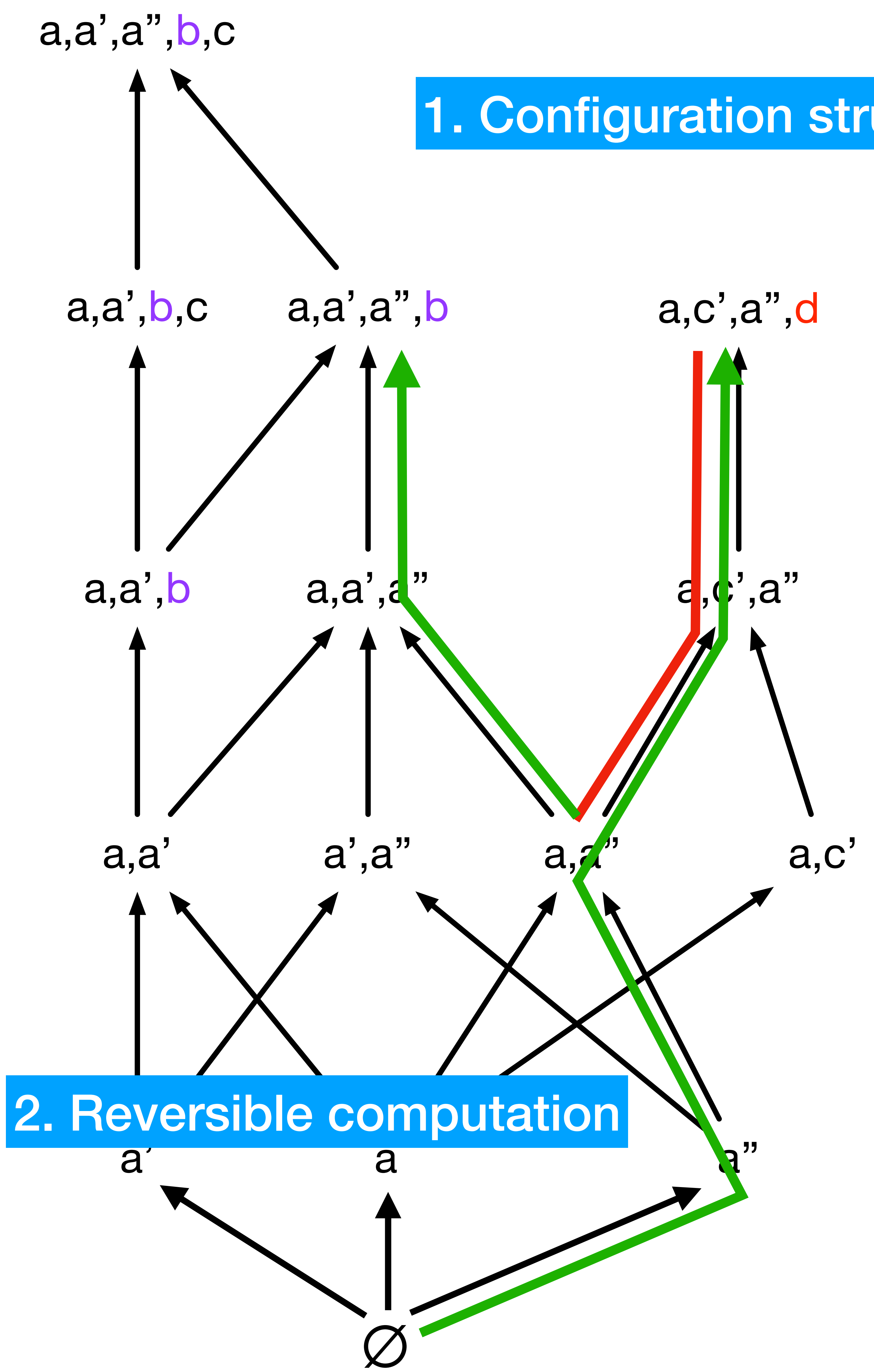

Logical characterisation

Time

$\emptyset$

$\{a, a'\} < b \ \wedge b < c$

$a' \ \# \ c' \quad a < c'$

$\{a'', c'\} < d$

Time

$\{a, c', a'', d\}$

↑

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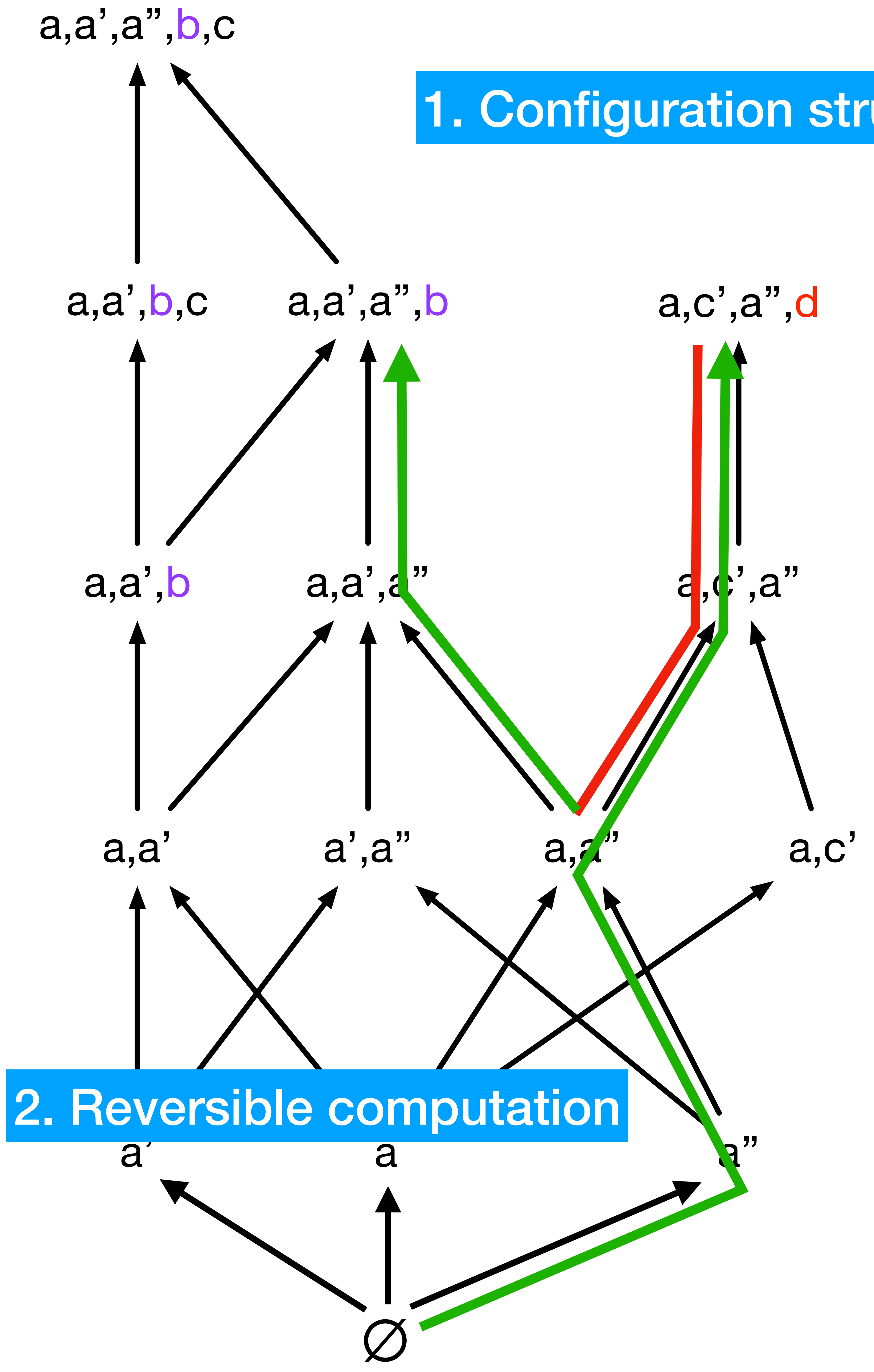

adjunction

Logical characterisation

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3. Prime event structures interpretation

Configuration structure

$\mathbb{C} = (E, X) \quad X \subseteq \mathscr{P}(E)$

Partial order

$\mathbb{C}^{\text{p.o}} = (X, \subseteq\,) \quad x \frown_{\mathbb{C}} y$ iff $\uparrow^{\mathbb{C}} \{x, y\} \neq \varnothing$

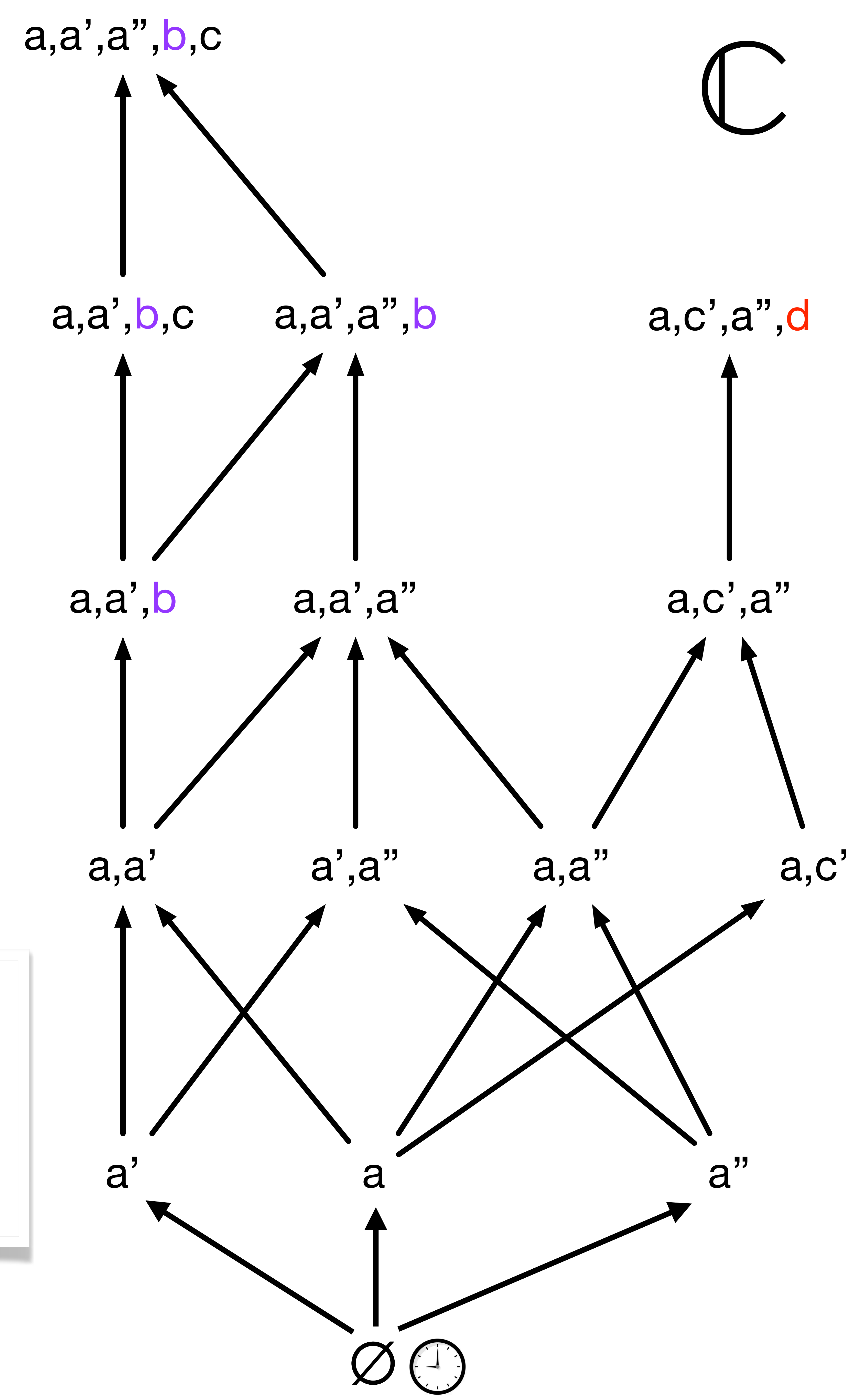
Atomic

$\mathbb{C}$ is *rooted* and *connected*

Definition 2.3 Let $\mathbb{C} \in \mathcal{C}_E$. For all finite $x \in \mathbb{C}$, we define the *residual* of $\mathbb{C}$ *after* x:

$$x \cdot \mathbb{C} := \langle E, \{z \in \mathcal{P}(E) \mid \exists y \in \uparrow^{\mathbb{C}}\{x\} : z = y \backslash x\}\rangle$$

where $y \backslash x := \{a \in y \mid a \notin x\}$ is the classical set difference.



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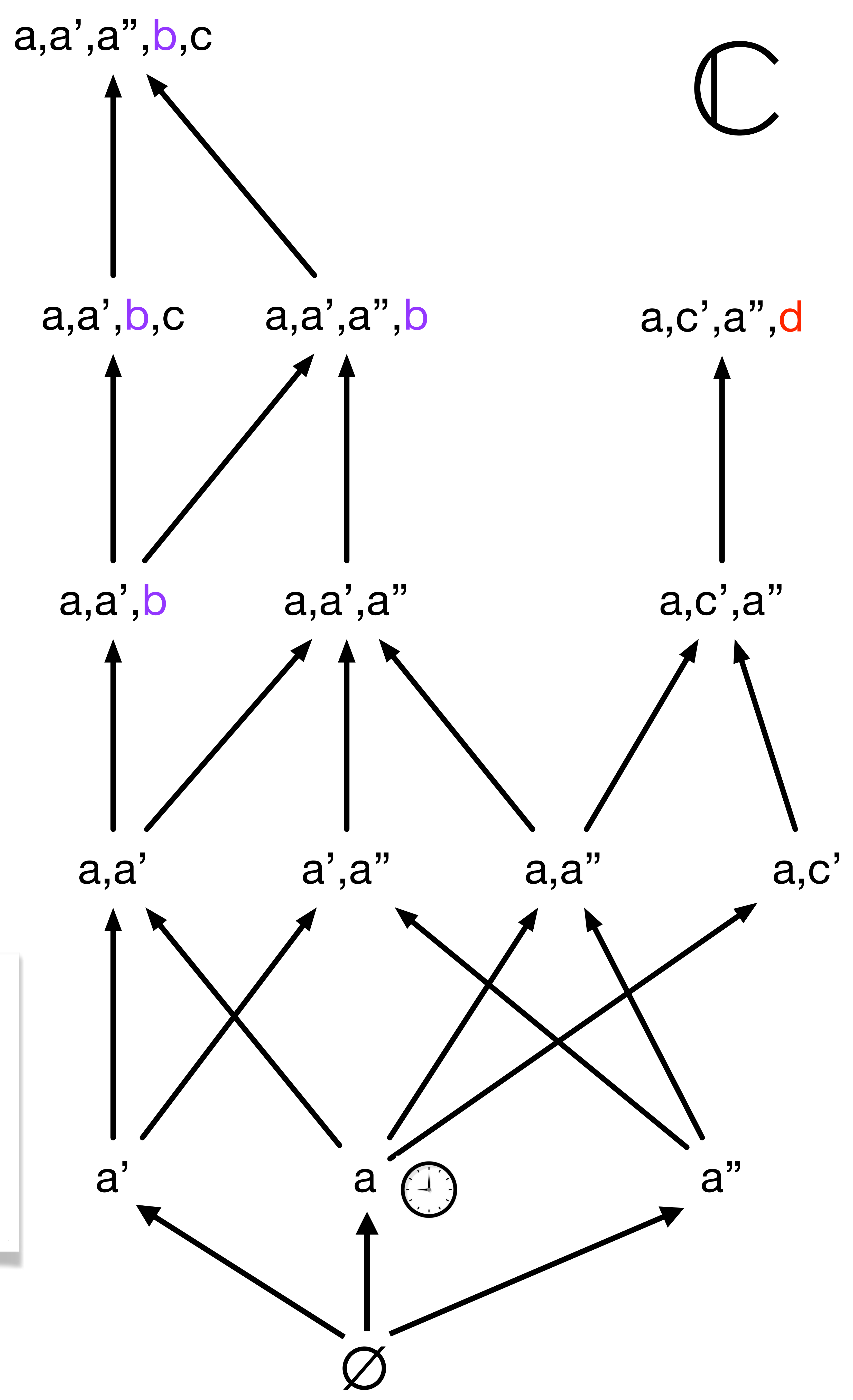

$$\{a\} \cdot \mathbb{C}$$

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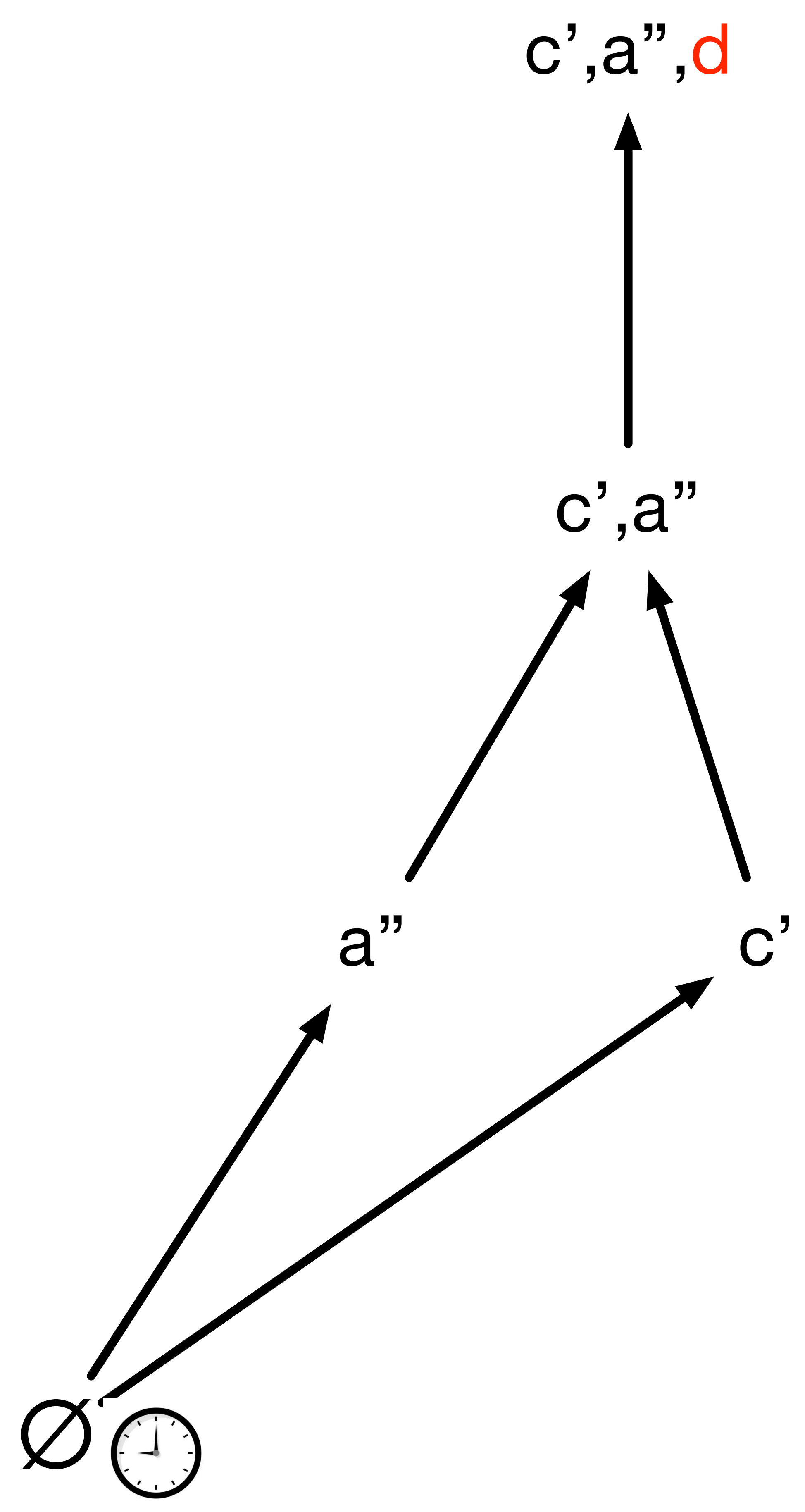

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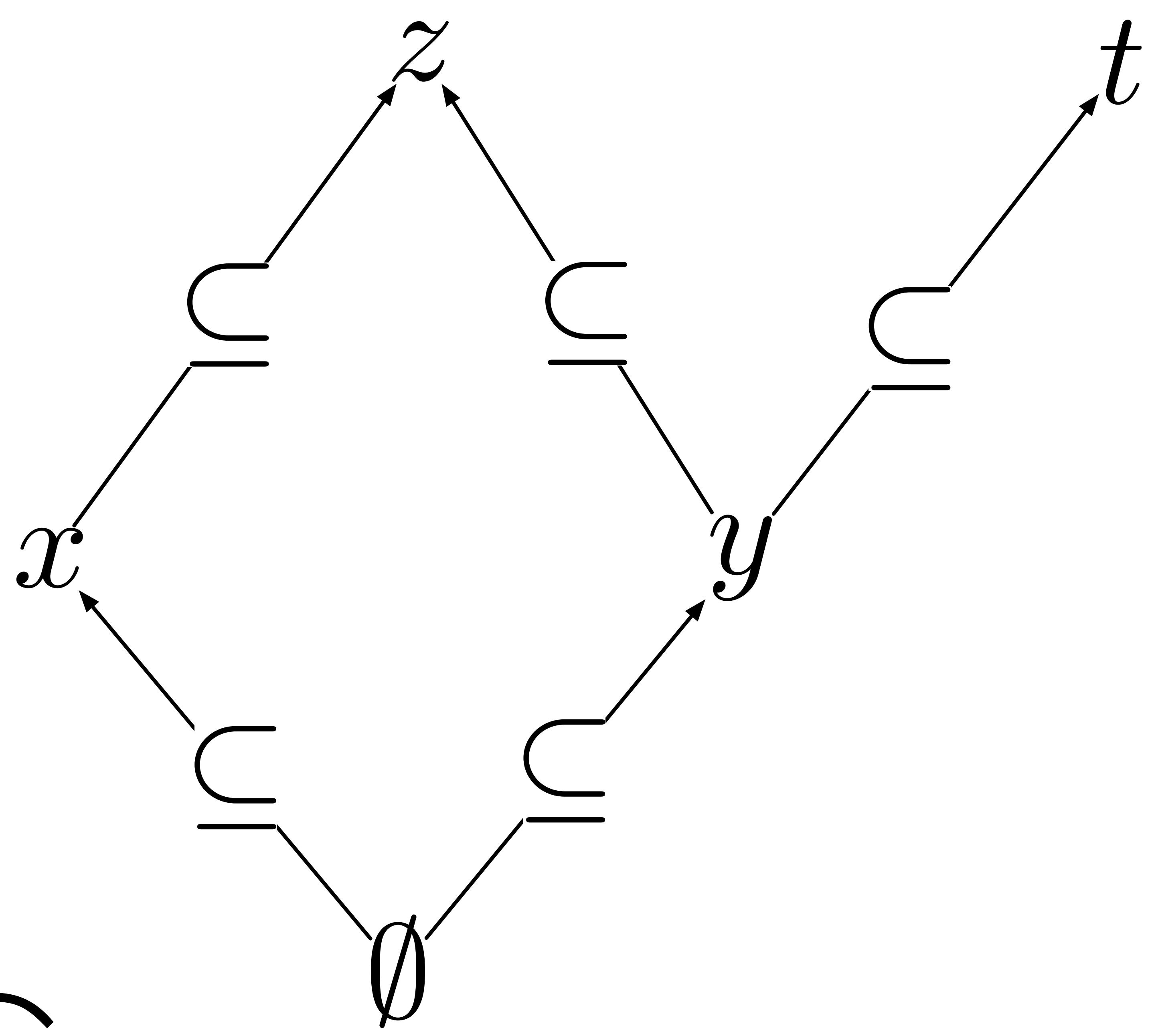
Forward computation

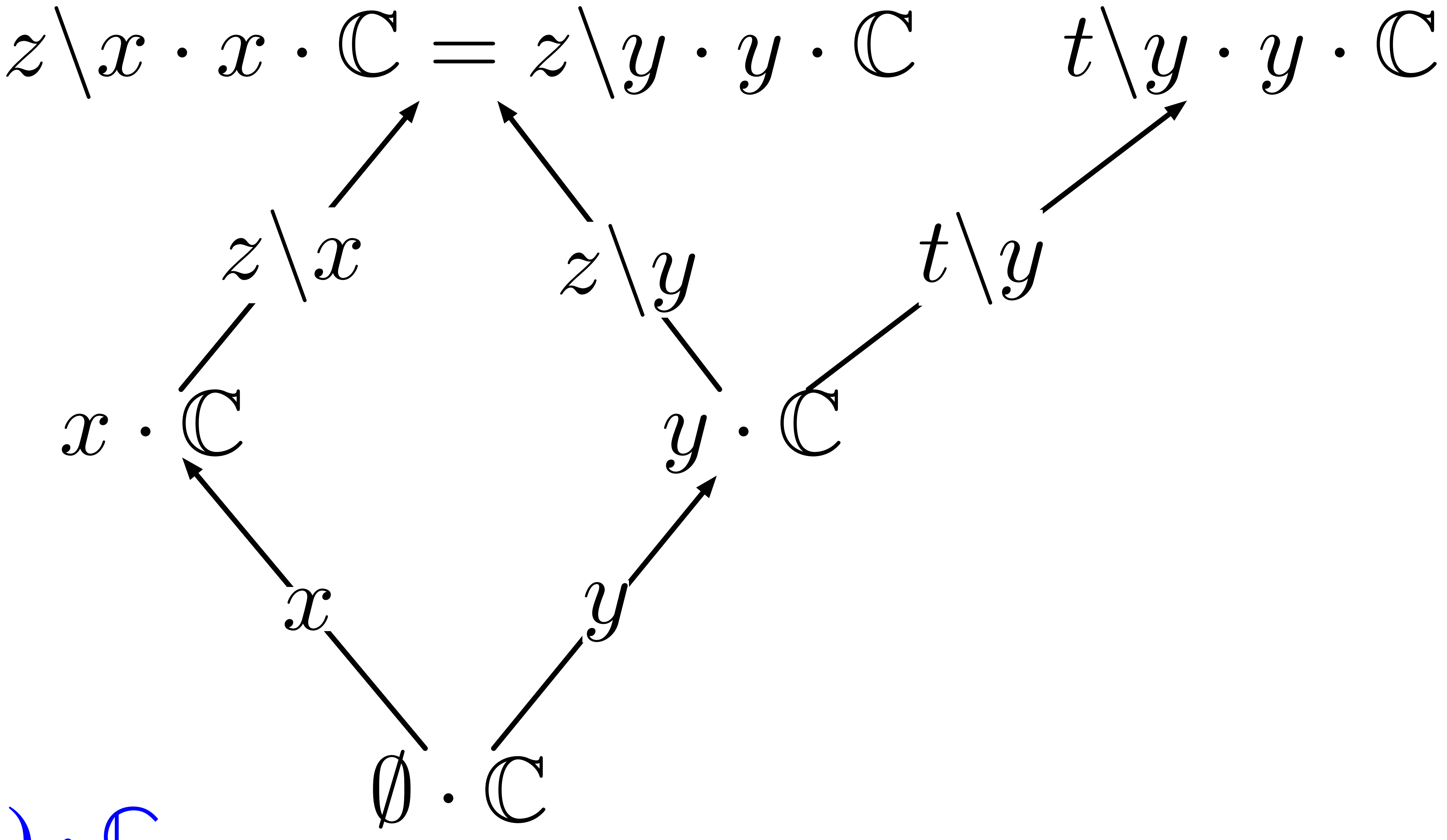


$\mathbb{C}$

+ residuation

$x_0 \cdot x_1 \cdot \mathbb{C} = (x_0 \cup x_1) \cdot \mathbb{C}$

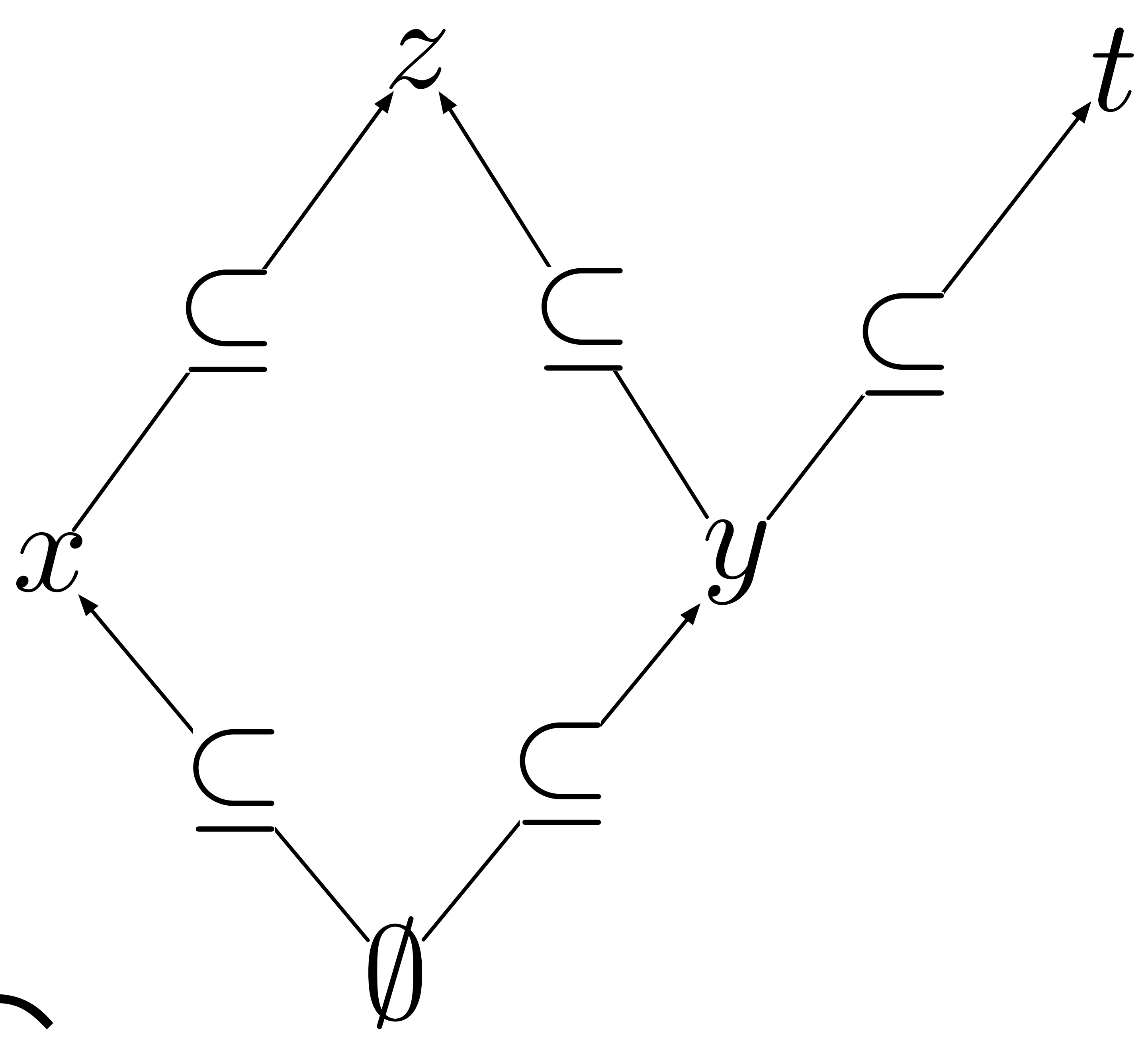
$\varnothing \cdot \mathbb{C} = \mathbb{C}$

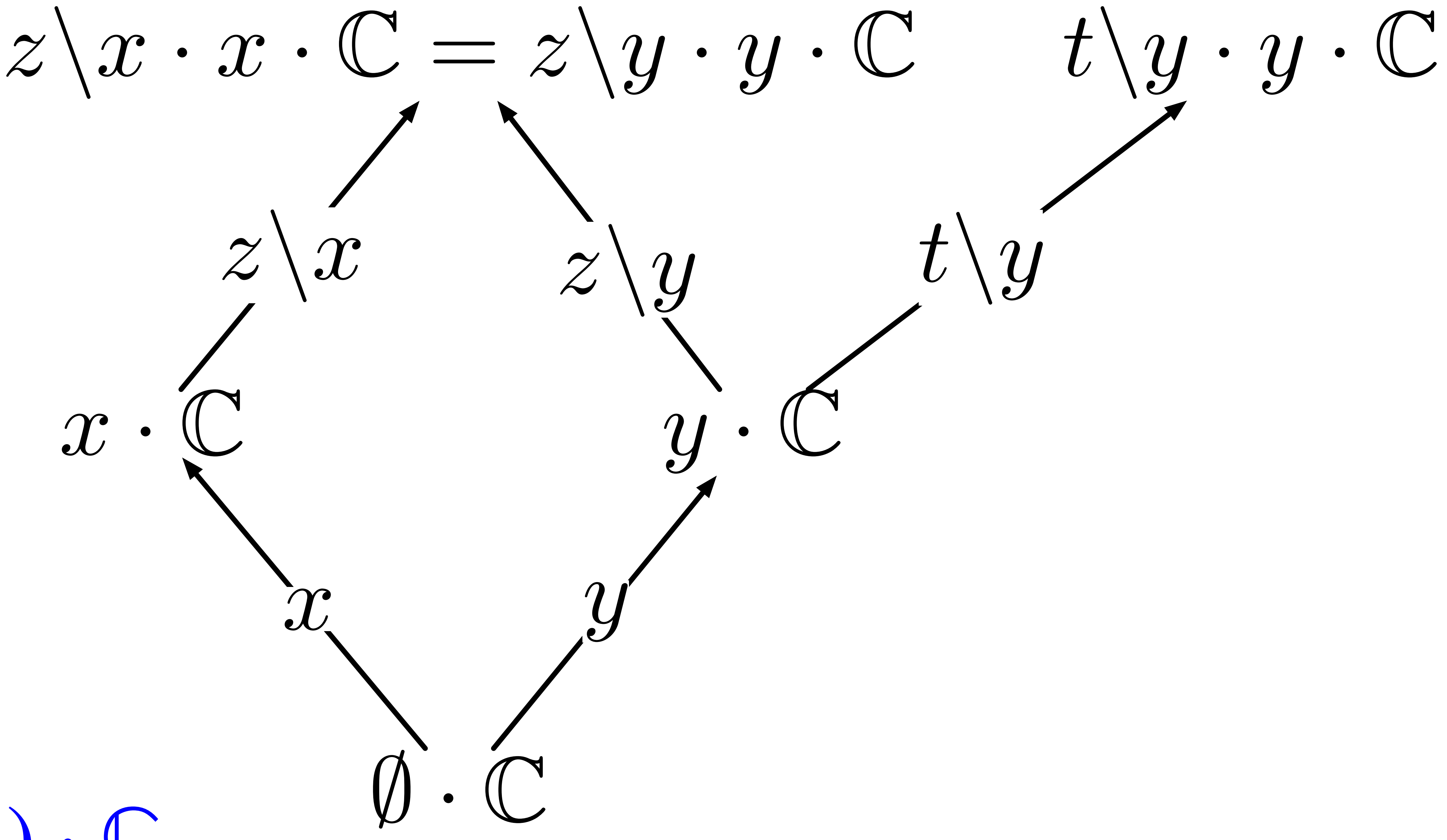


Labelled transition system
as a monoid action

Forward computation

Possible futures are partially ordered





$\mathbb{C}$

+ residuation

$x_0 \cdot x_1 \cdot \mathbb{C} = (x_0 \cup x_1) \cdot \mathbb{C}$

$\emptyset \cdot \mathbb{C} = \mathbb{C}$

Labelled transition system
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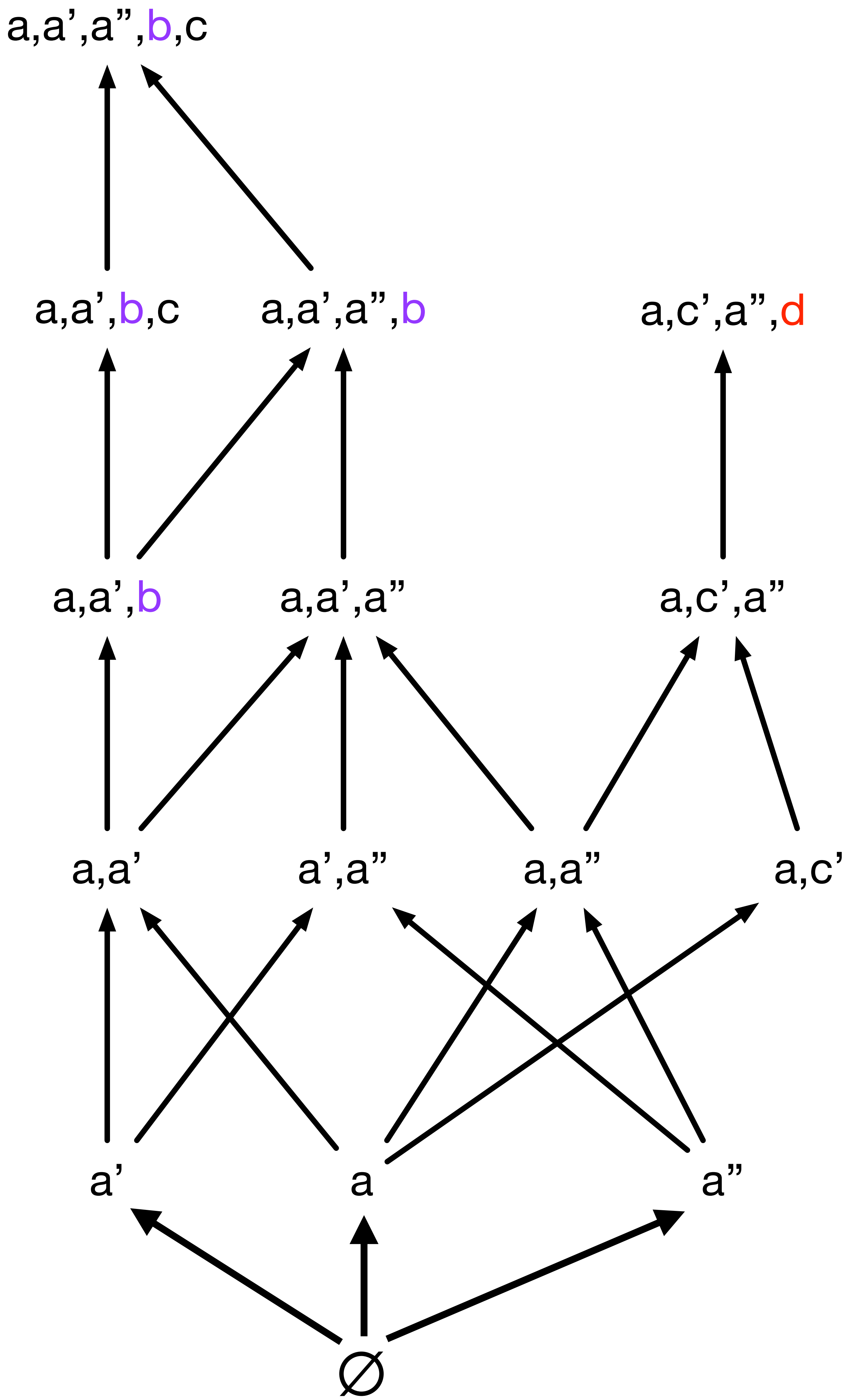
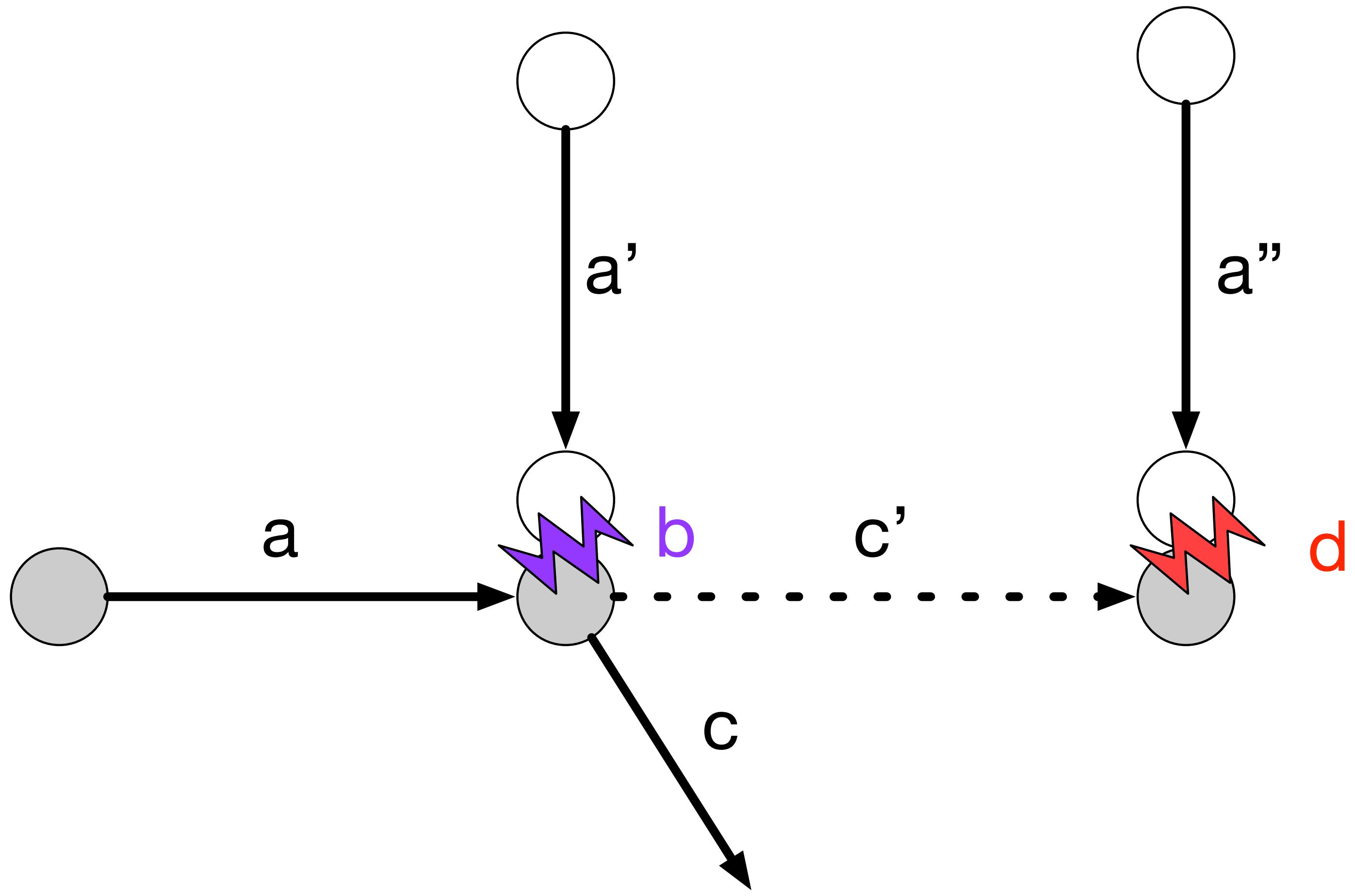
Prime Event Structures



Is this structure engendered by a causality and conflict relation amongst events?



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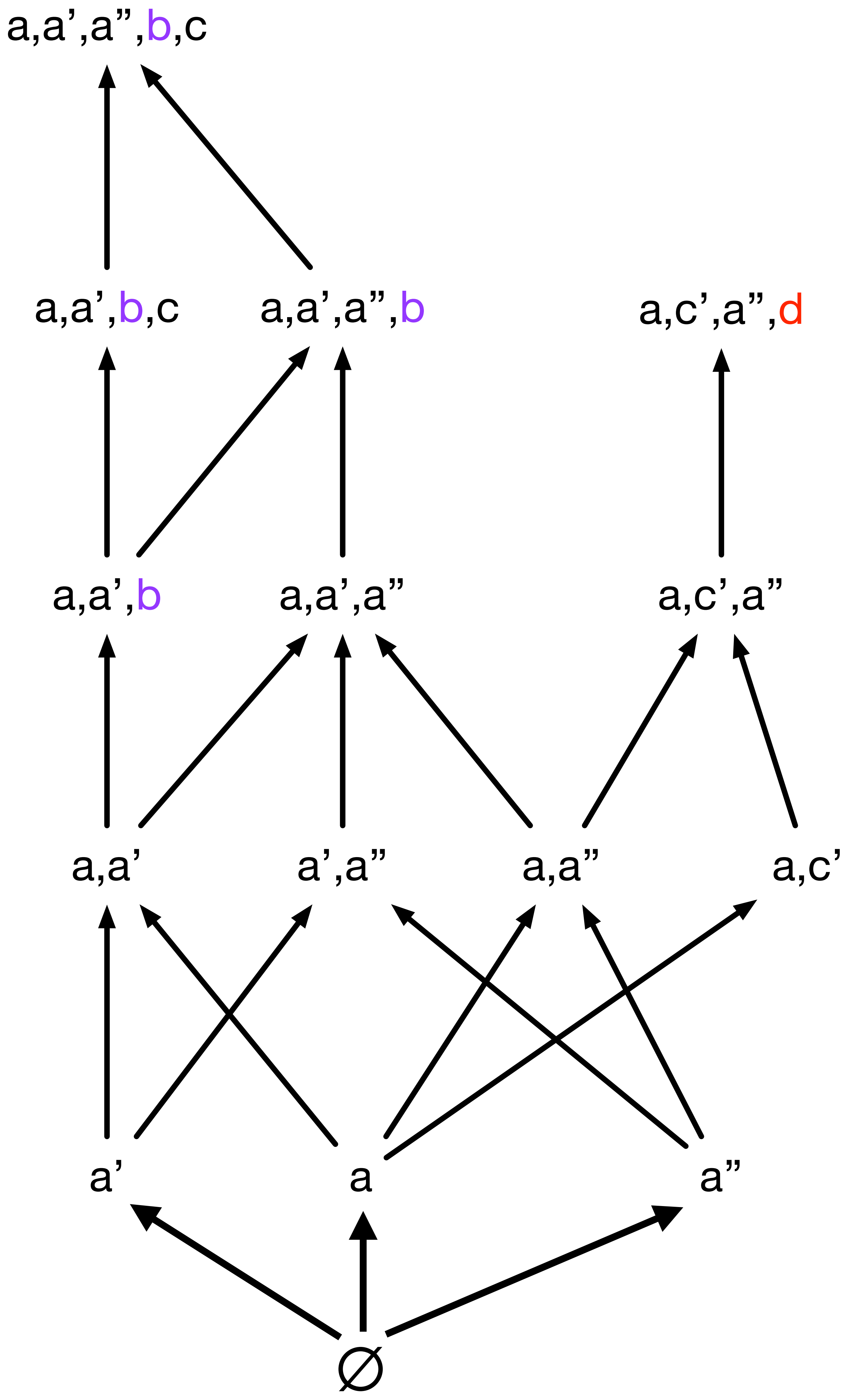

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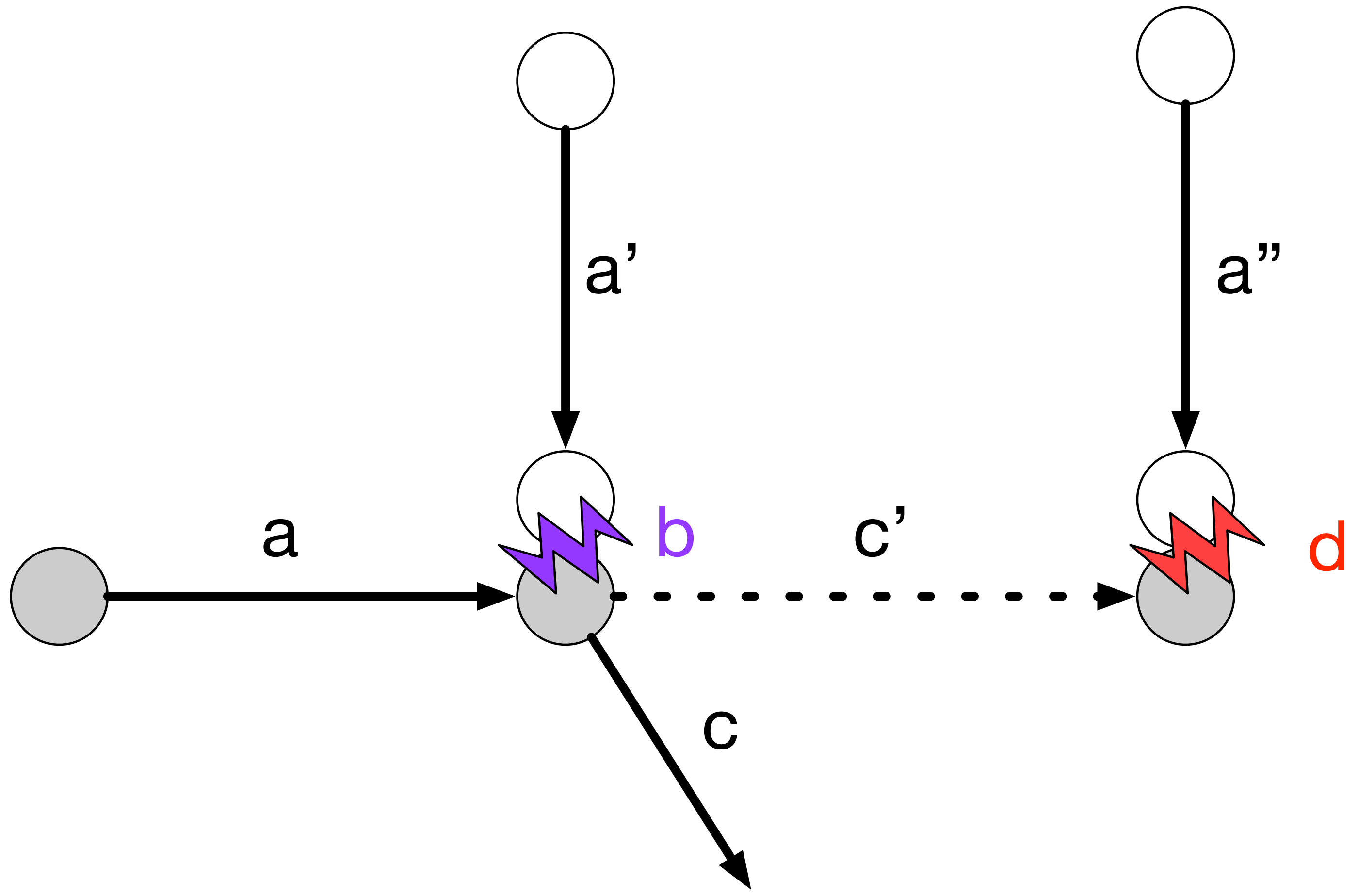
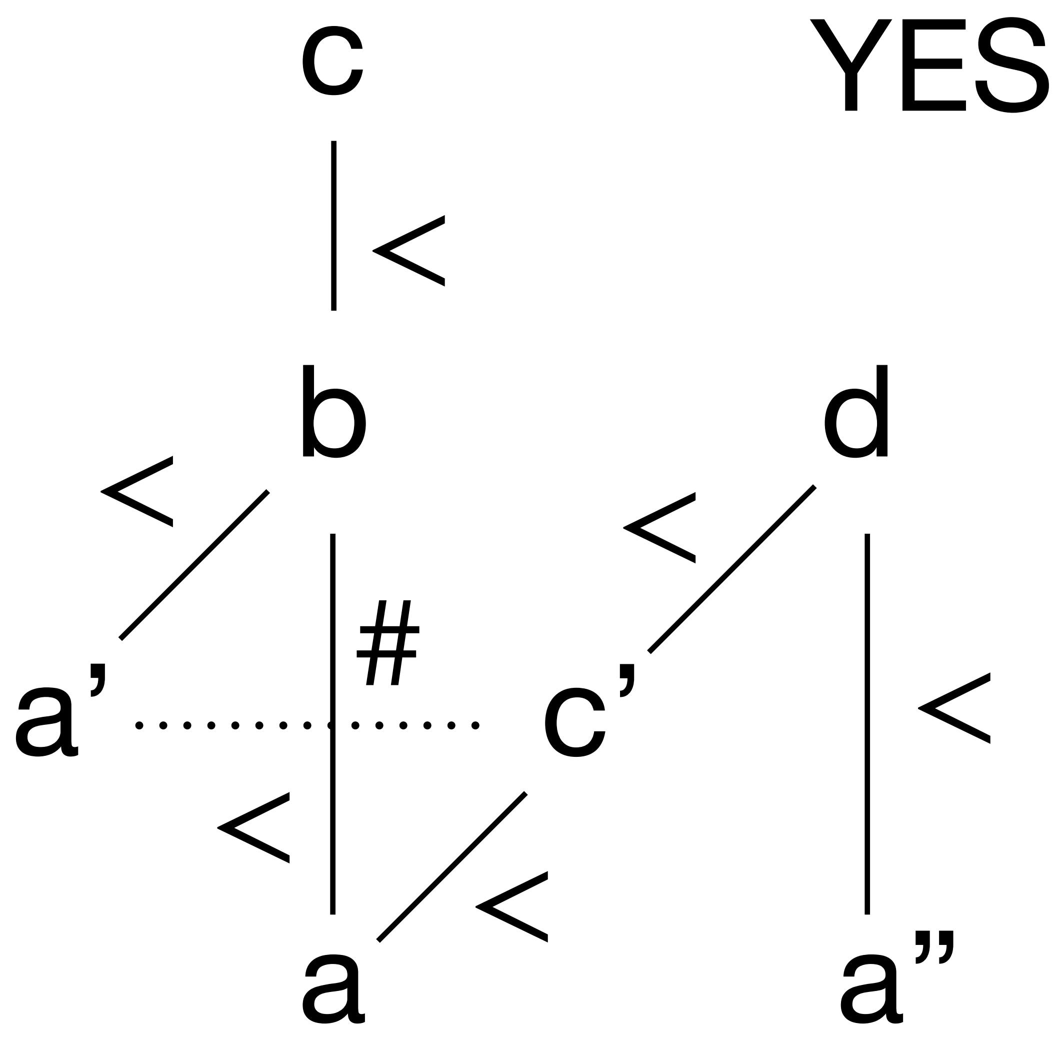
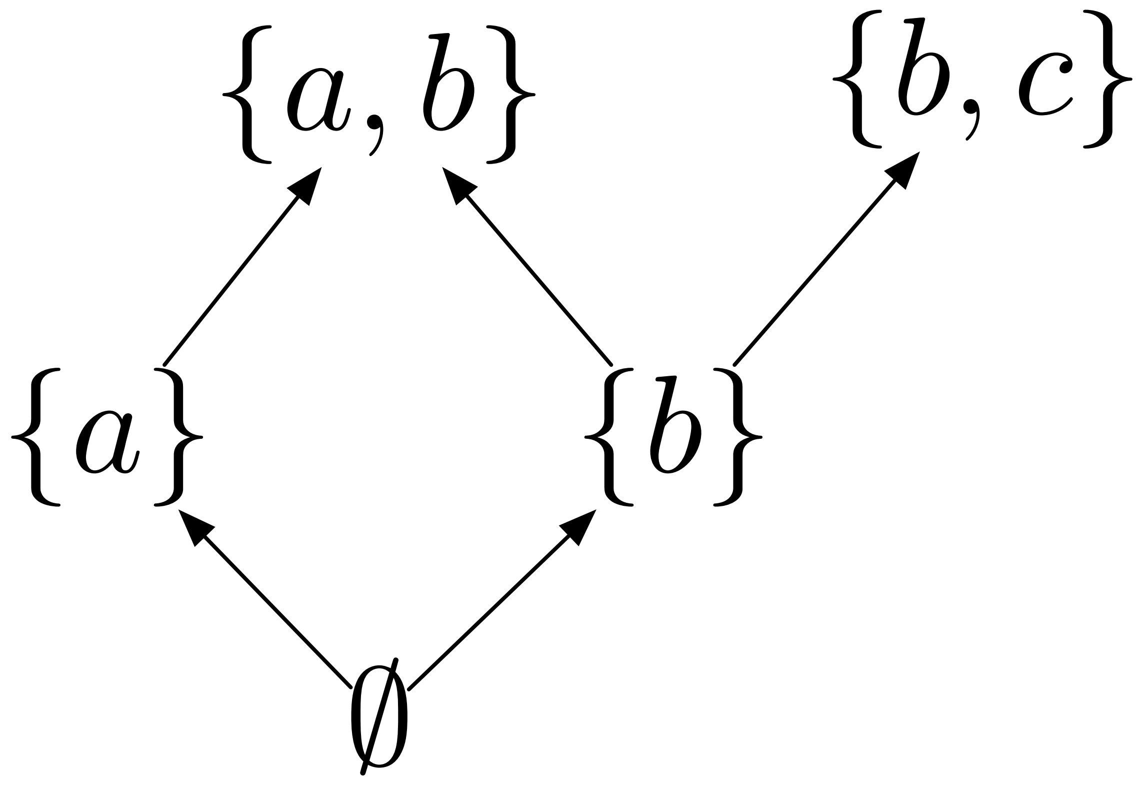




YES!

… but it is not true in general…

Adjunction [Winskel 82]

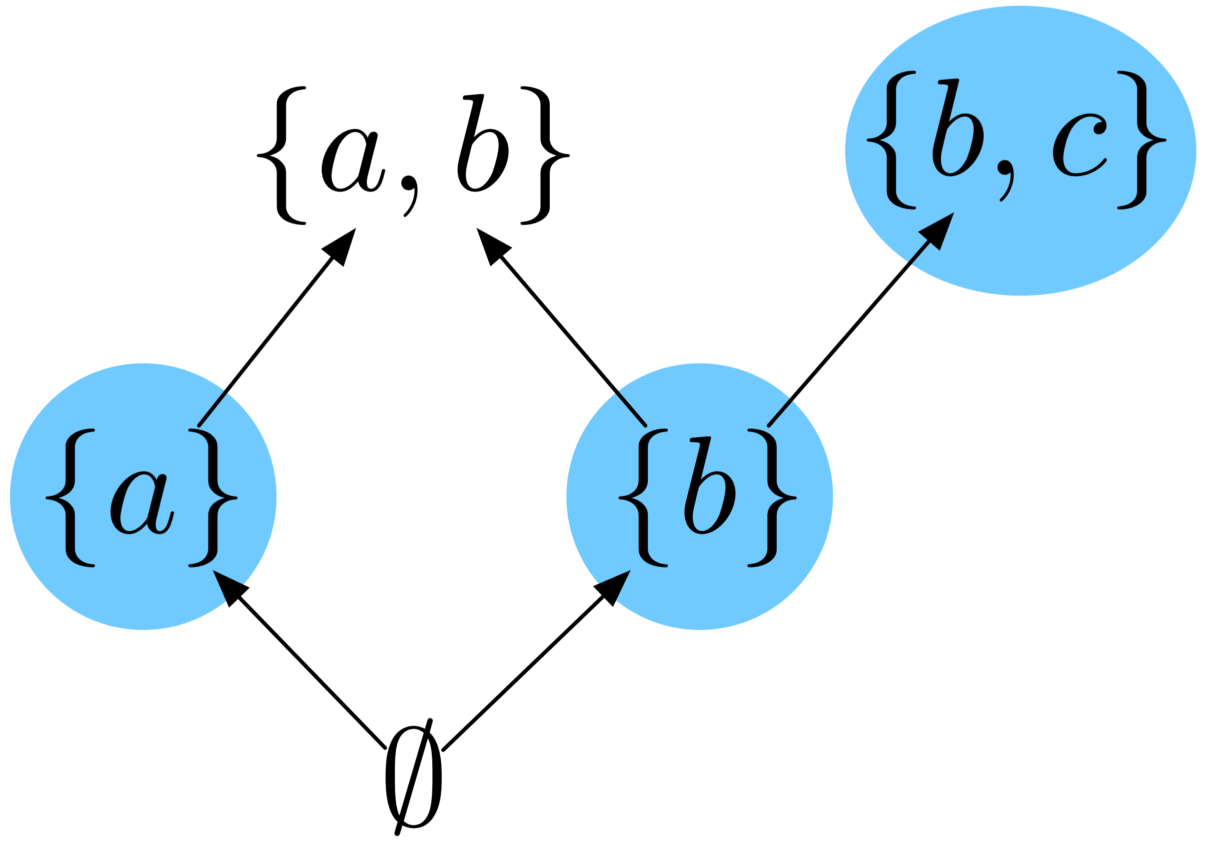


$\mathbb{C}$

Adjunction [Winskel 82]

Prime elements:

$$p \in \mathsf{Pr}(\mathbb{C}) : p \sqsubseteq \bigsqcup X \implies p \sqsubseteq x \in X$$



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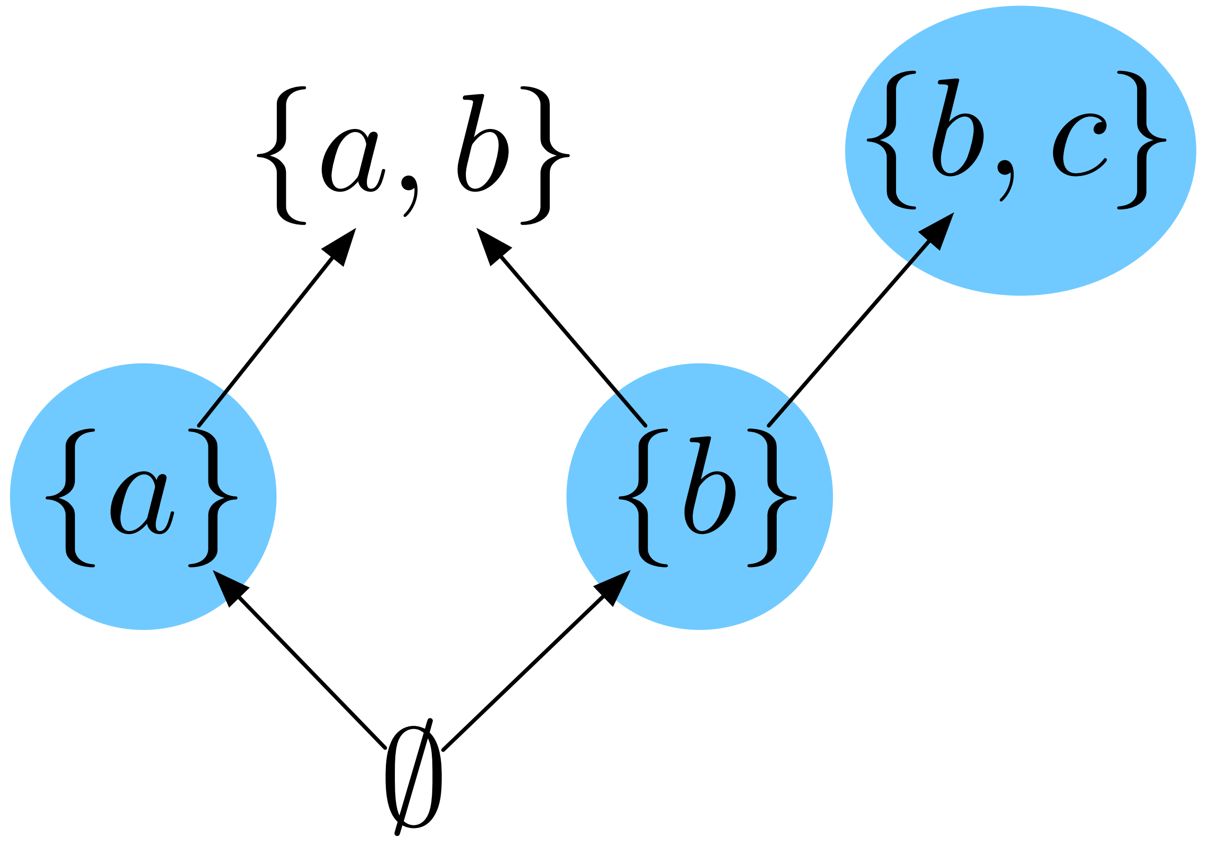

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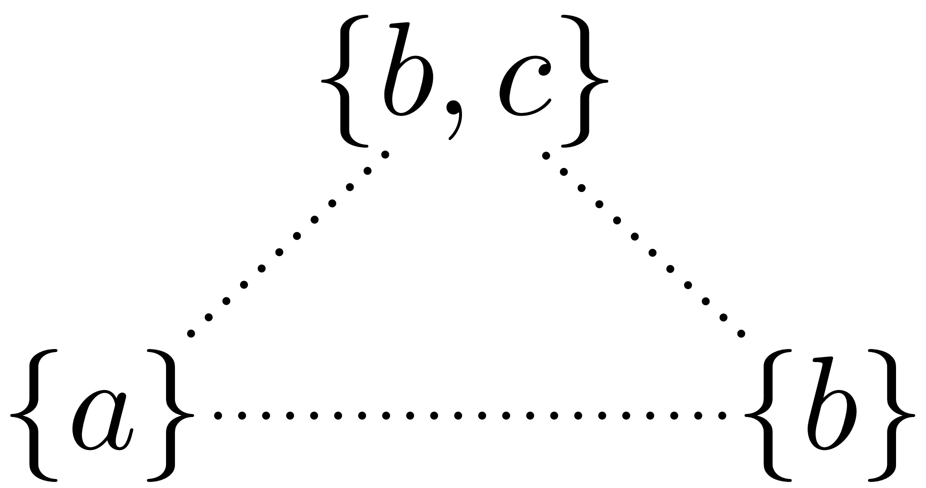

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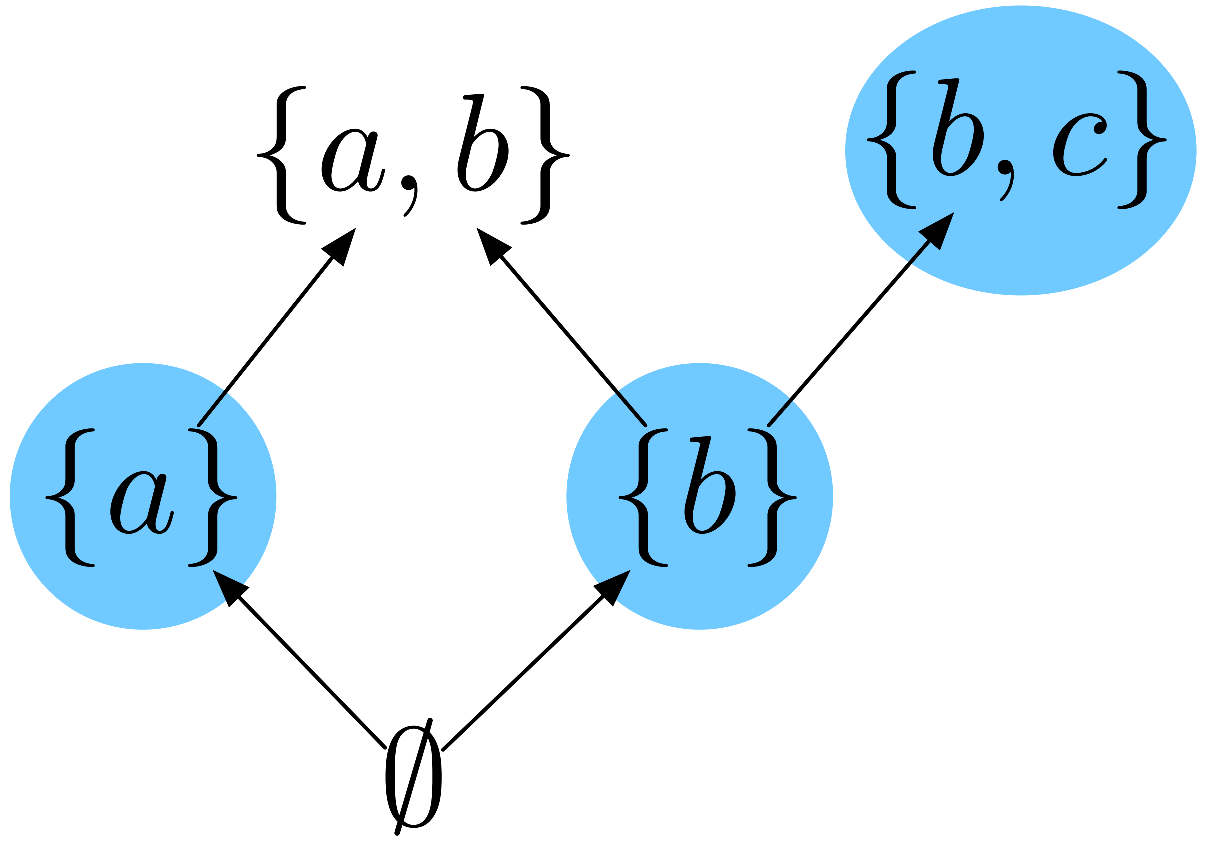
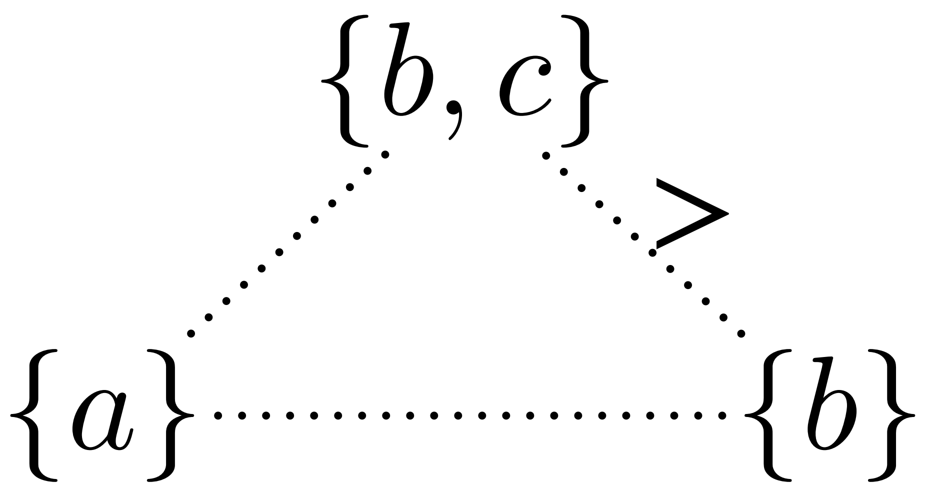


ℂ



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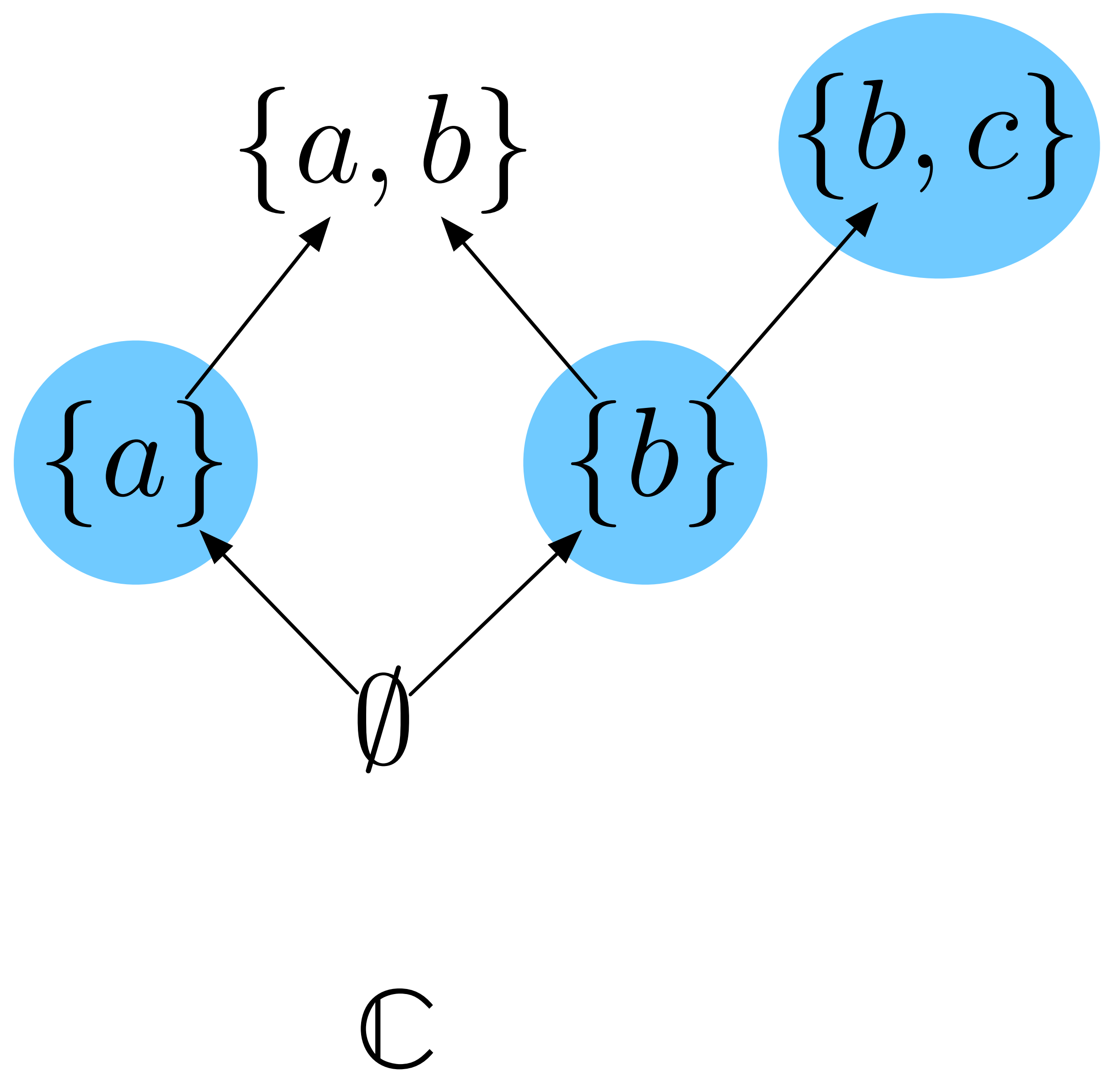

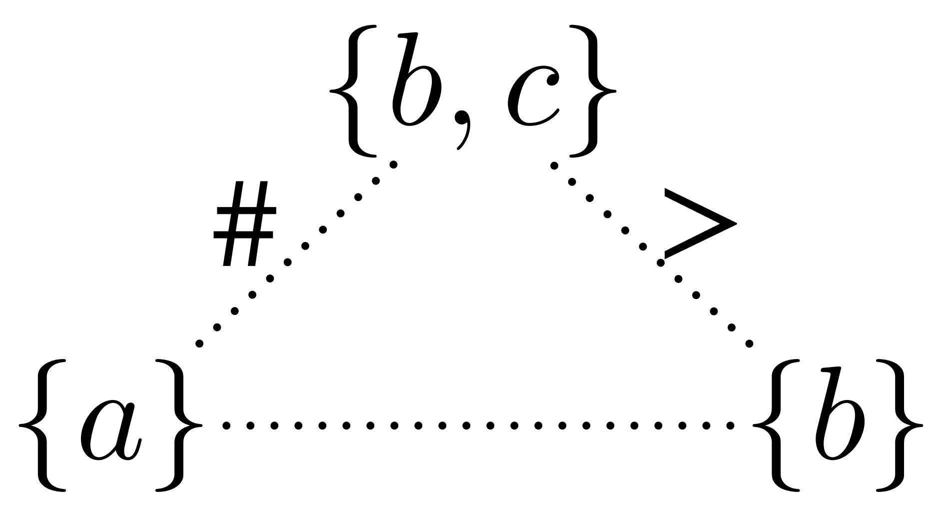

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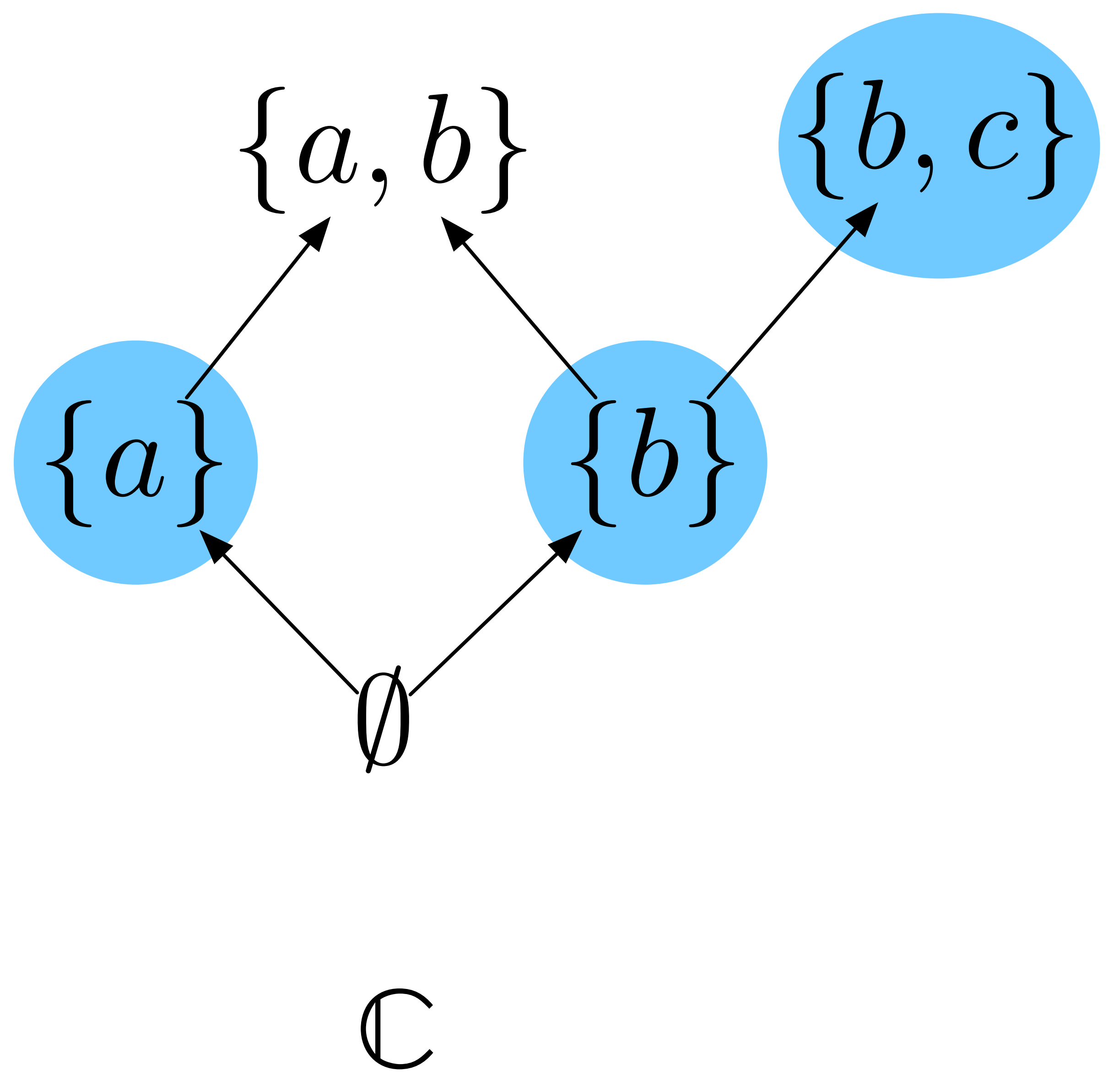

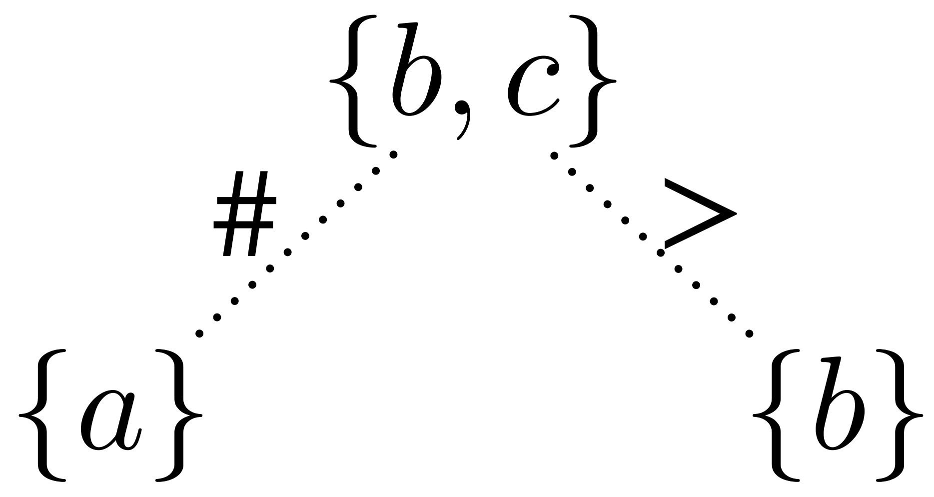

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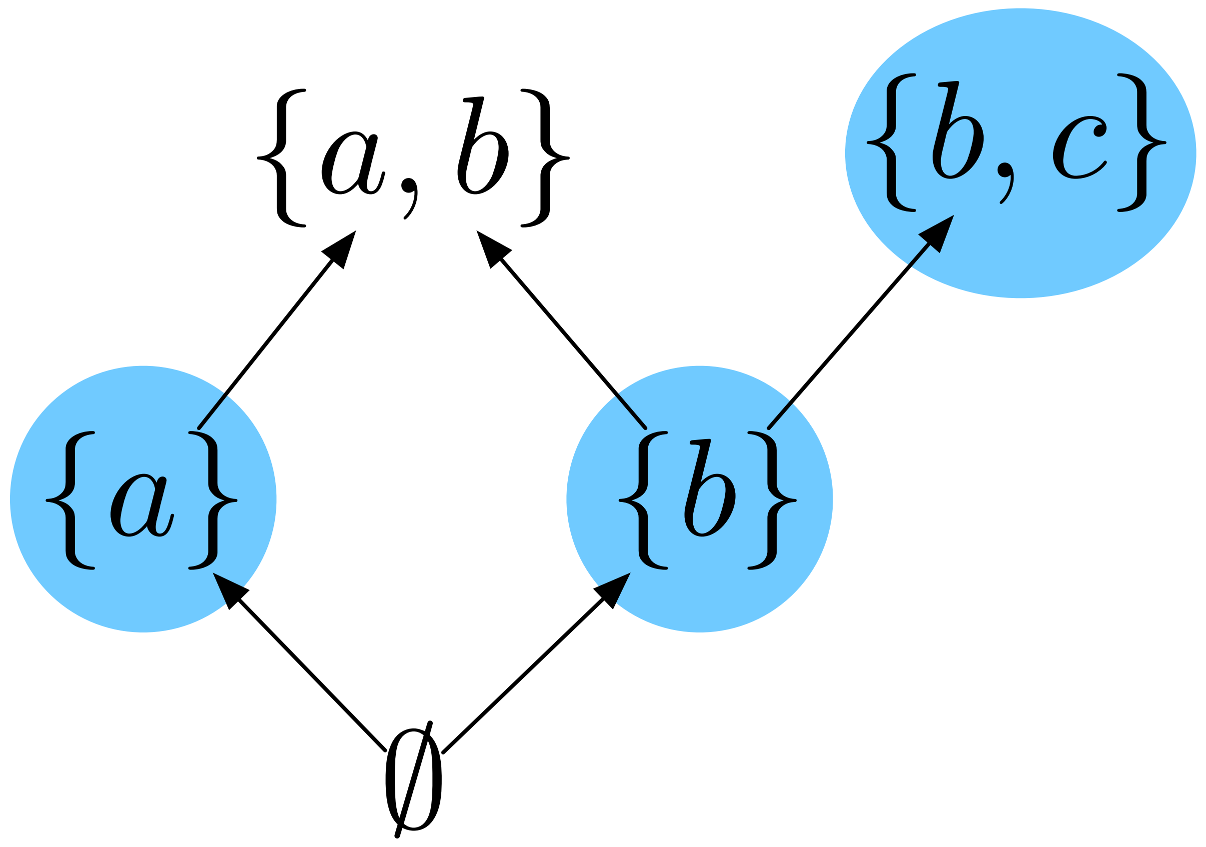
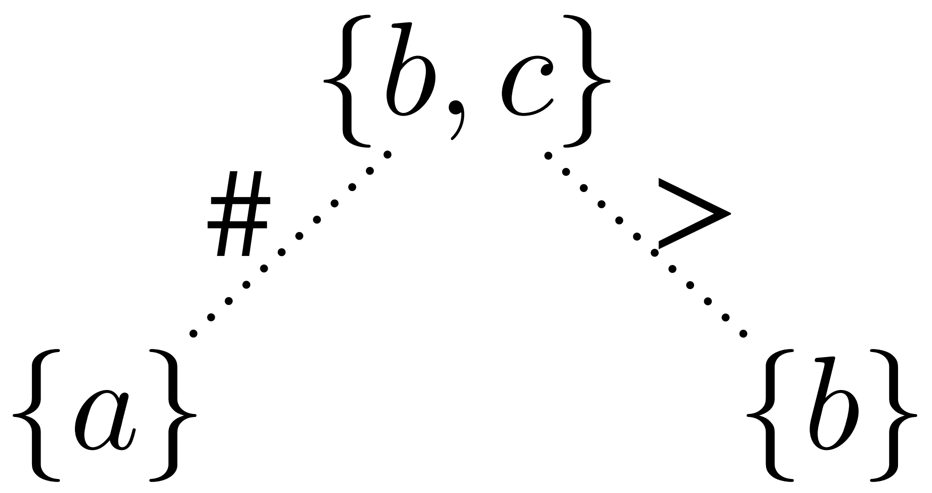


$\mathbb{C}$



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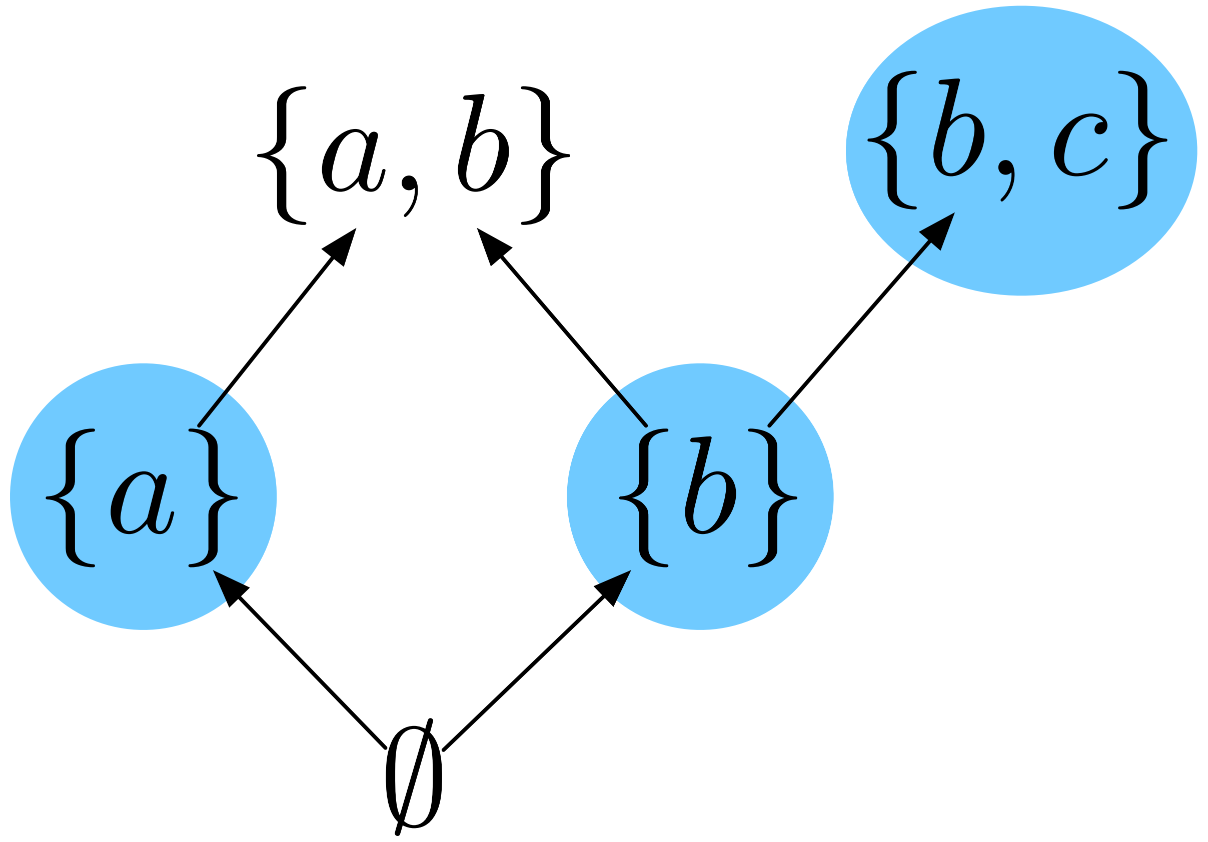

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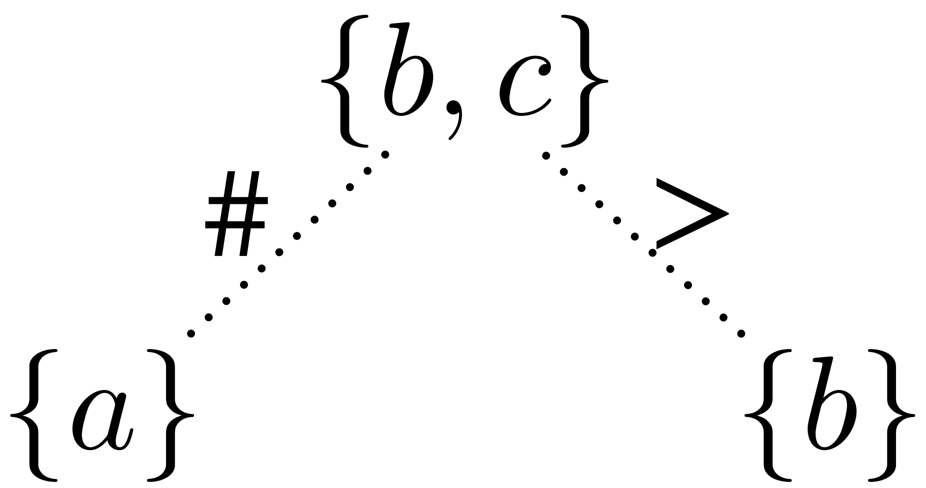

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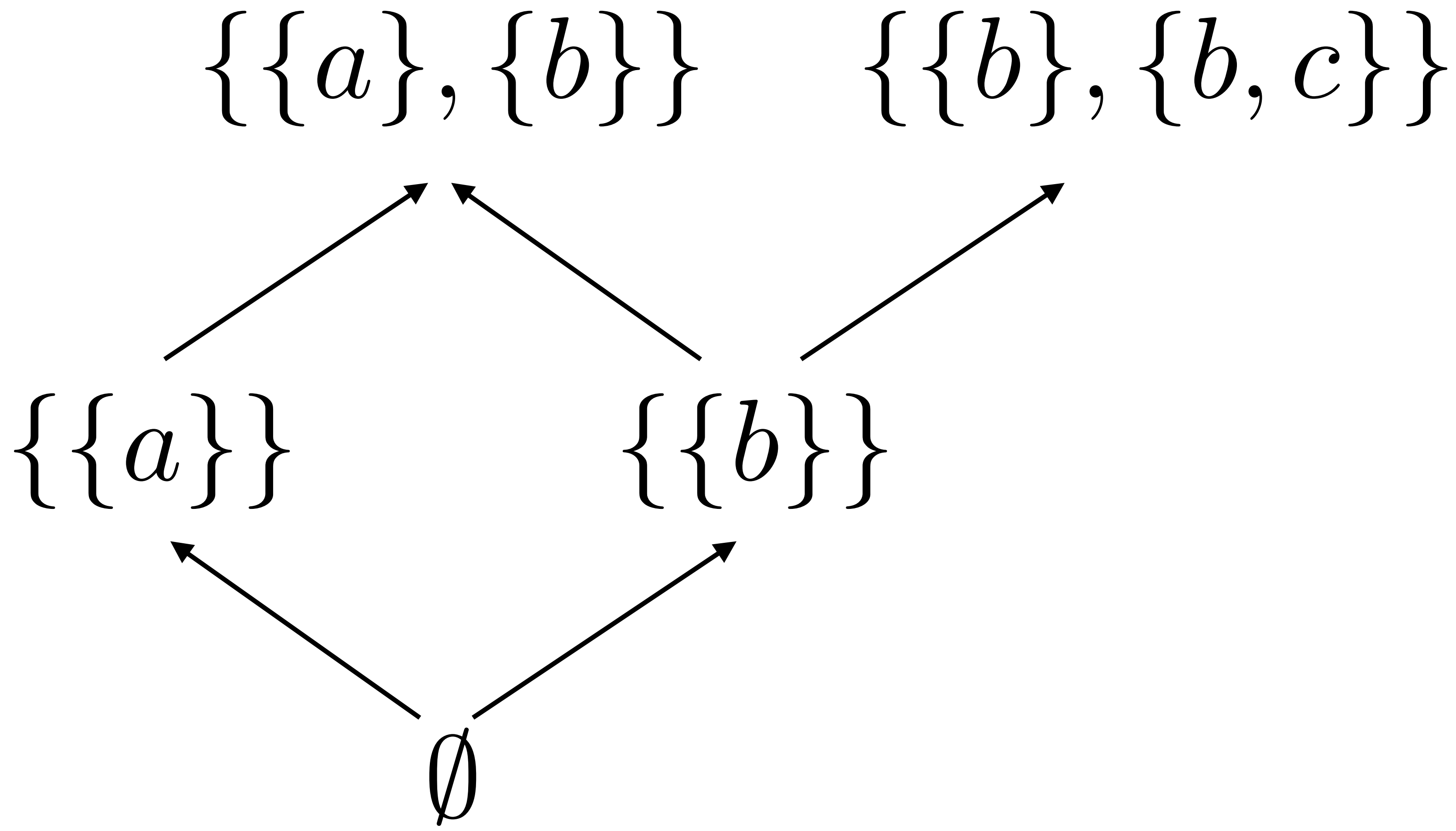

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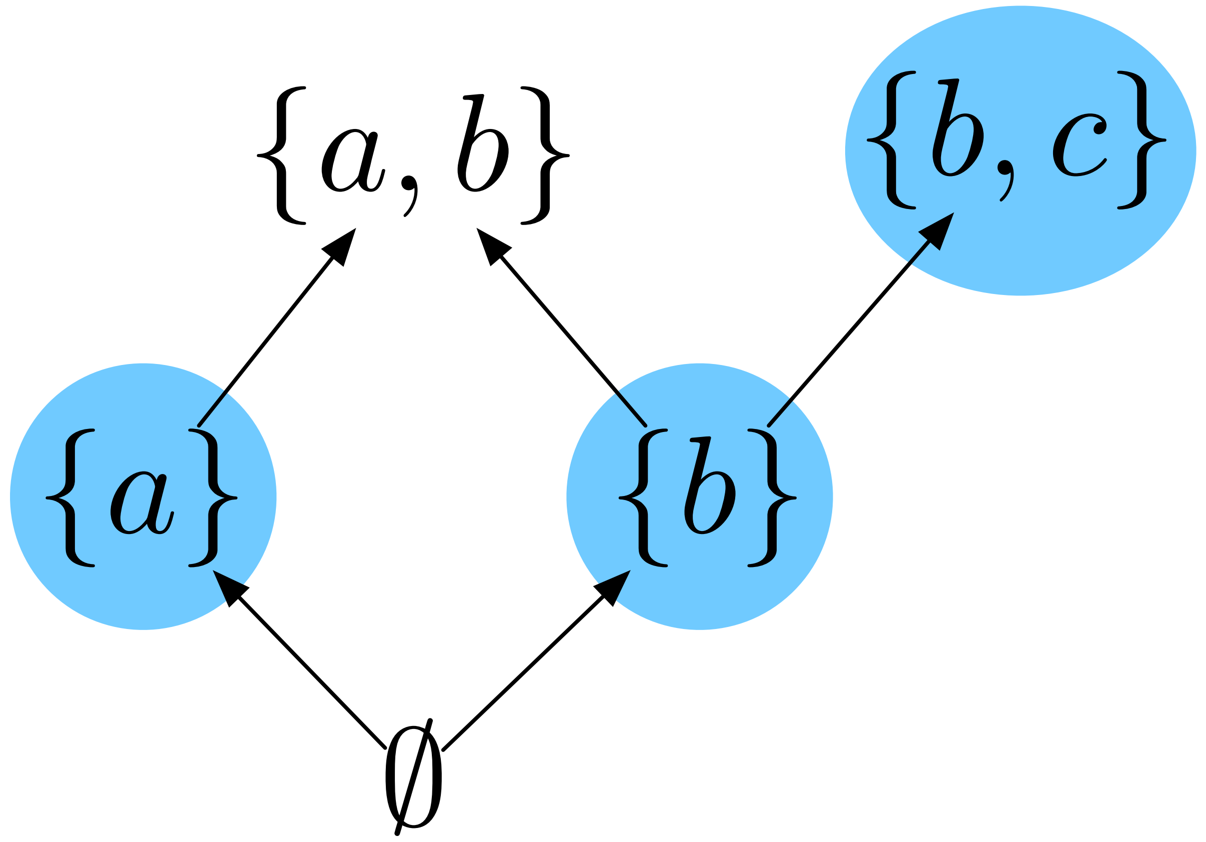

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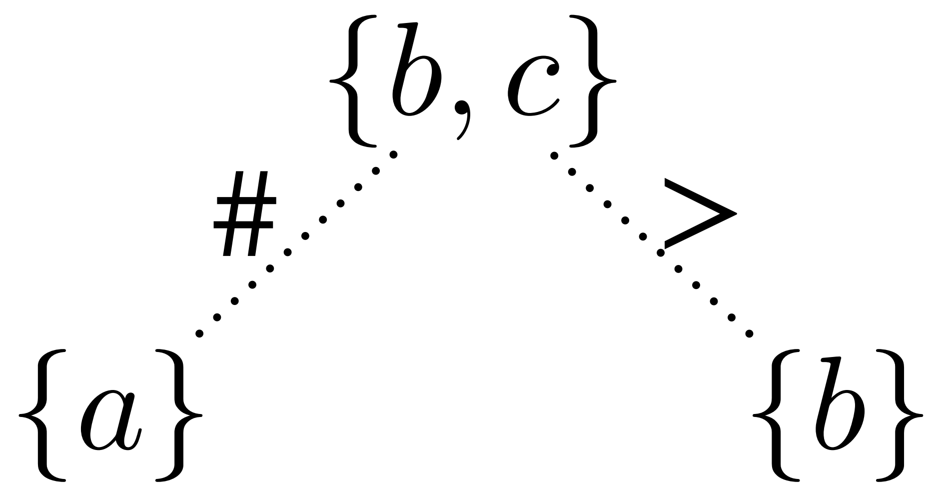

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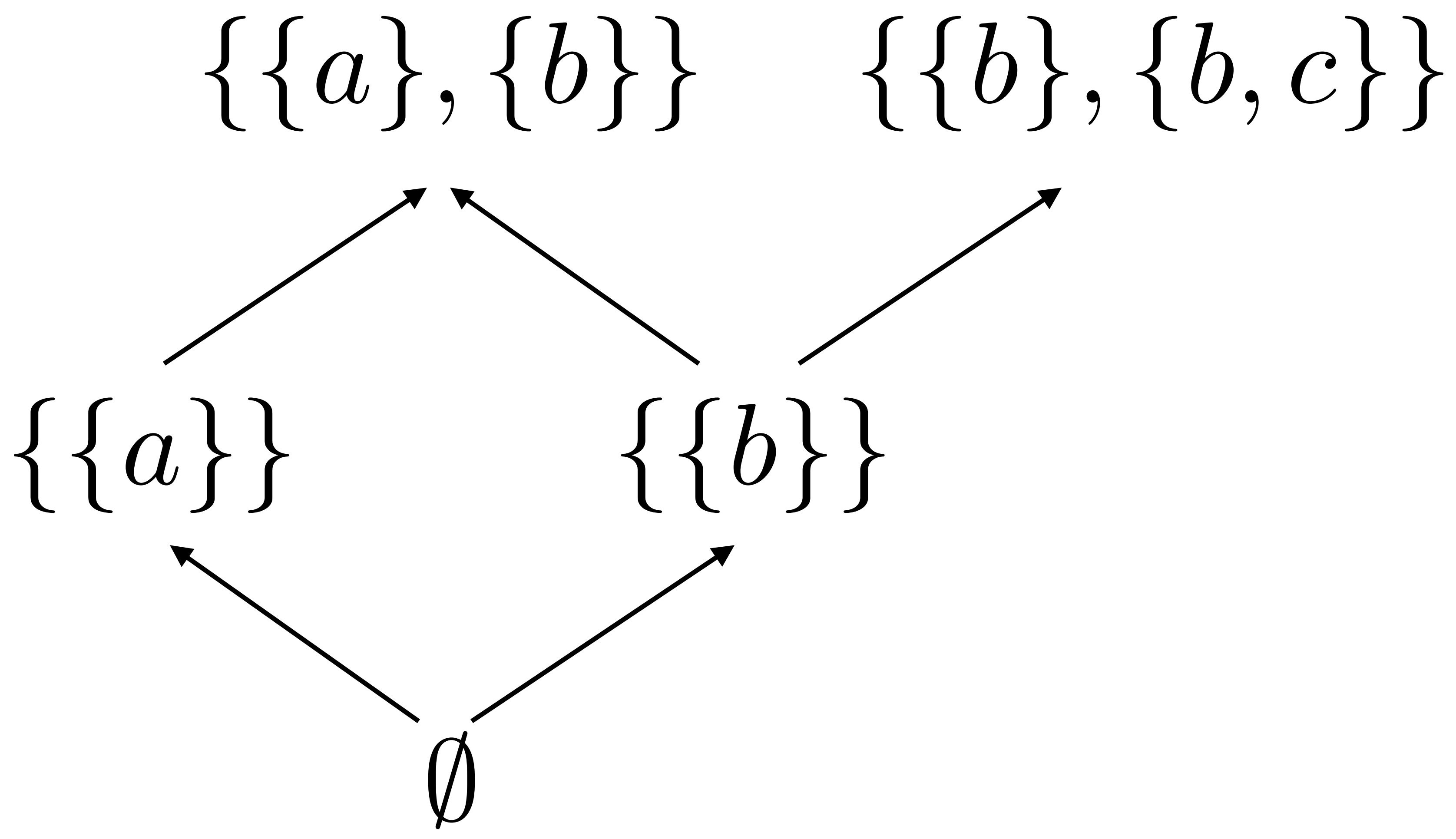
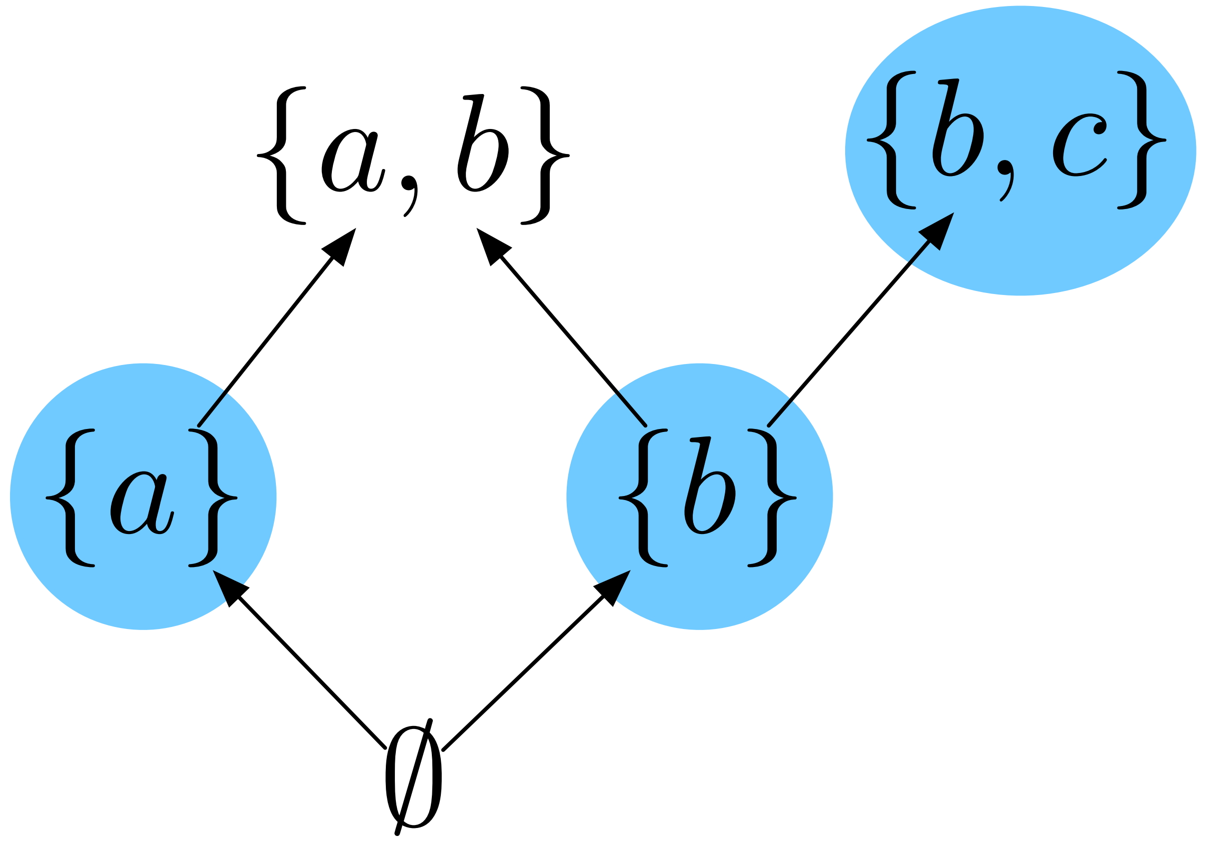


$\mathbb{C}' = \mathsf{Conf}(\mathbb{E})$

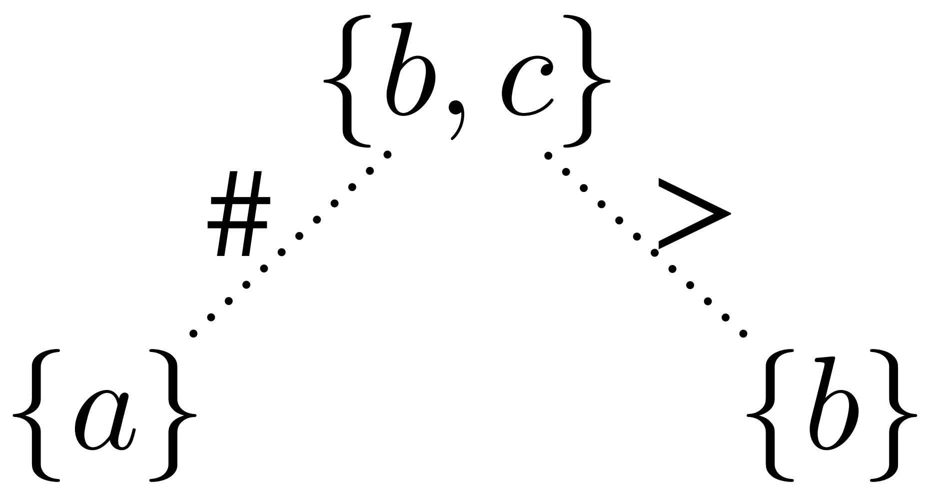
Adjunction [Winskel 82]

Prime elements:

$$p \in \mathsf{Pr}(\mathbb{C}) : p \sqsubseteq \bigsqcup X \implies p \sqsubseteq x \in X$$



$\mathbb{C}$ prime algebraic & coherent



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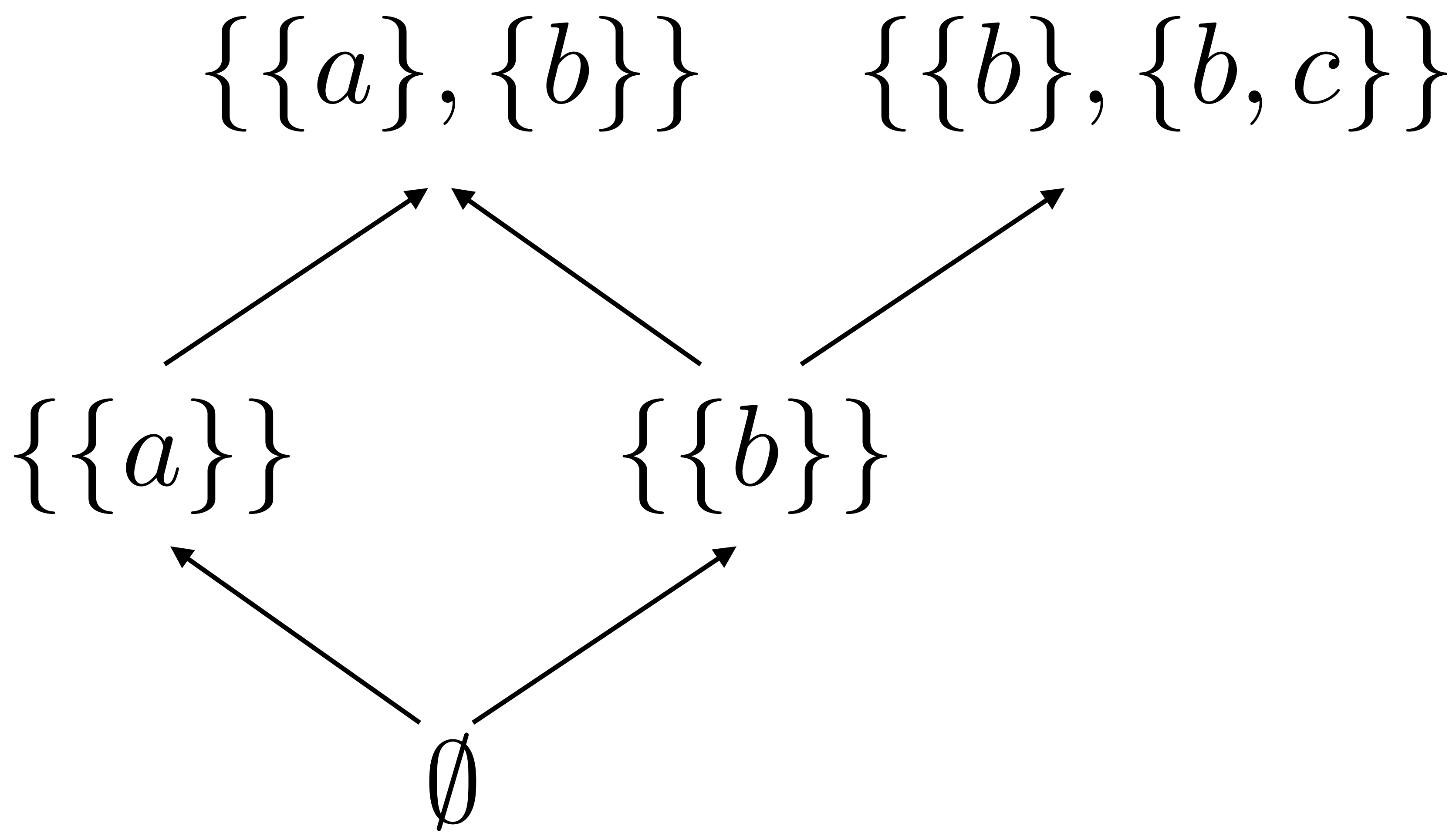

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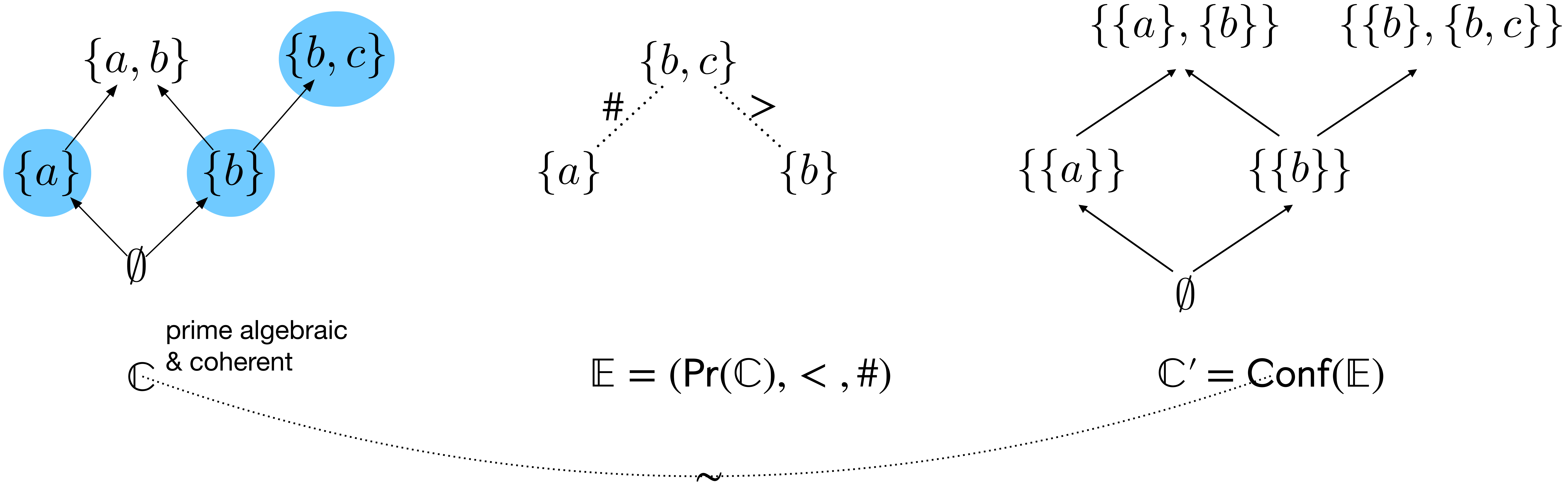
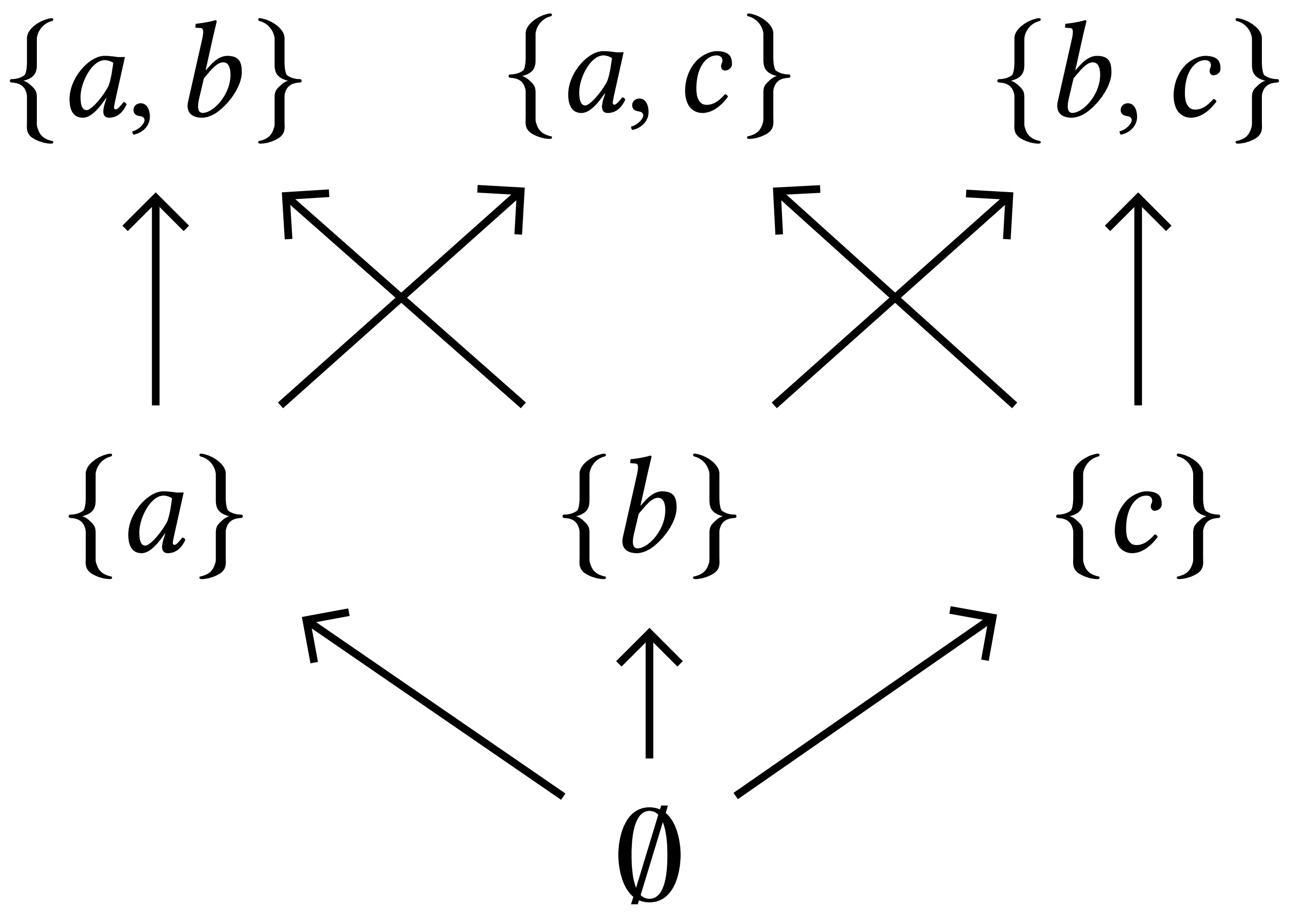
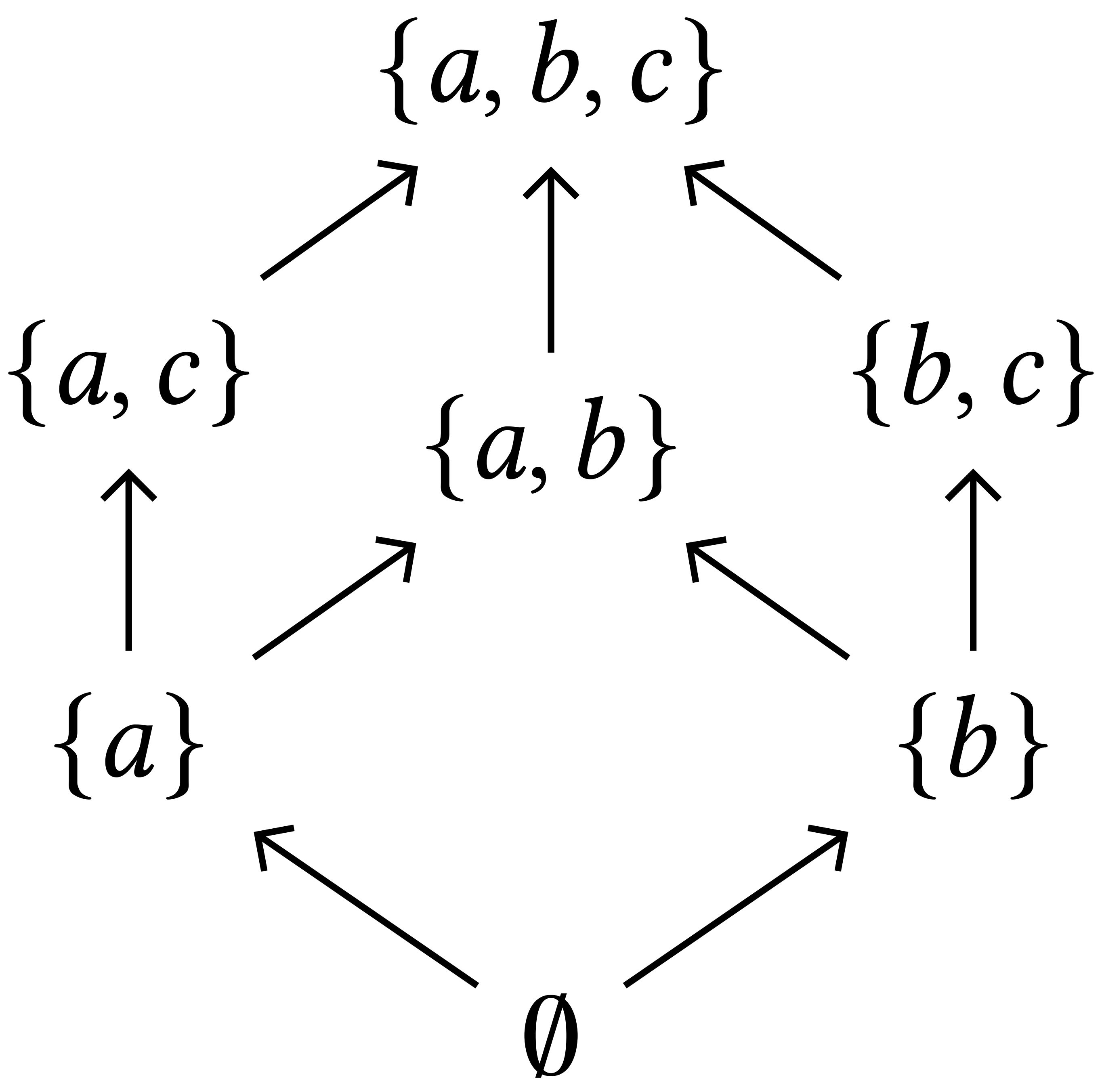


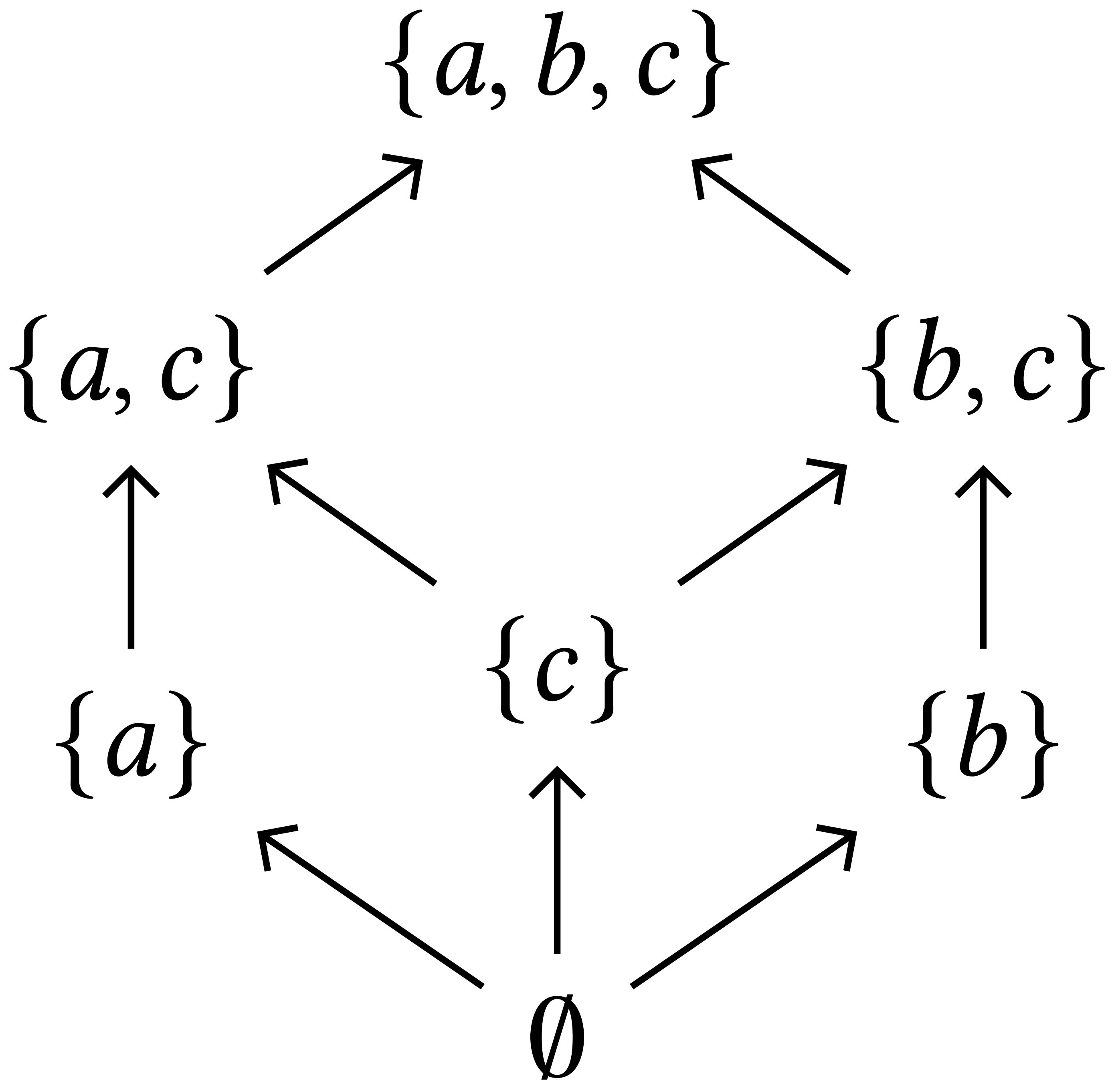
Stable configuration structures



$\mathbb{C}_0$

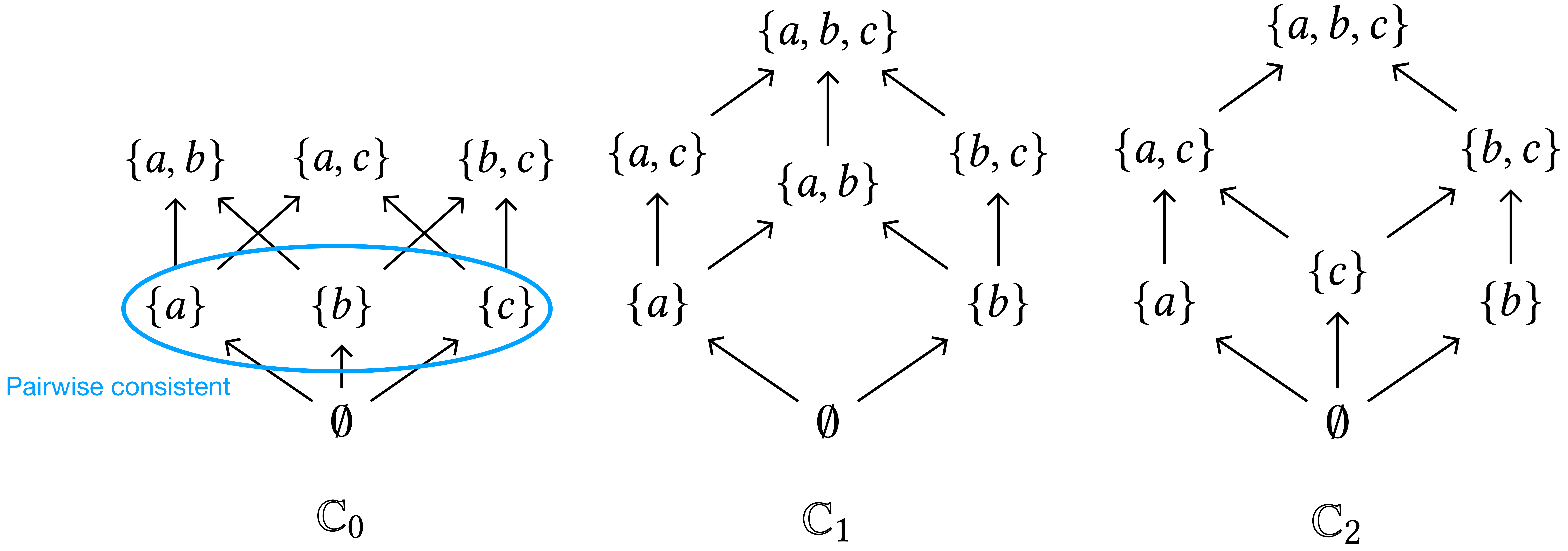


$\mathbb{C}_1$

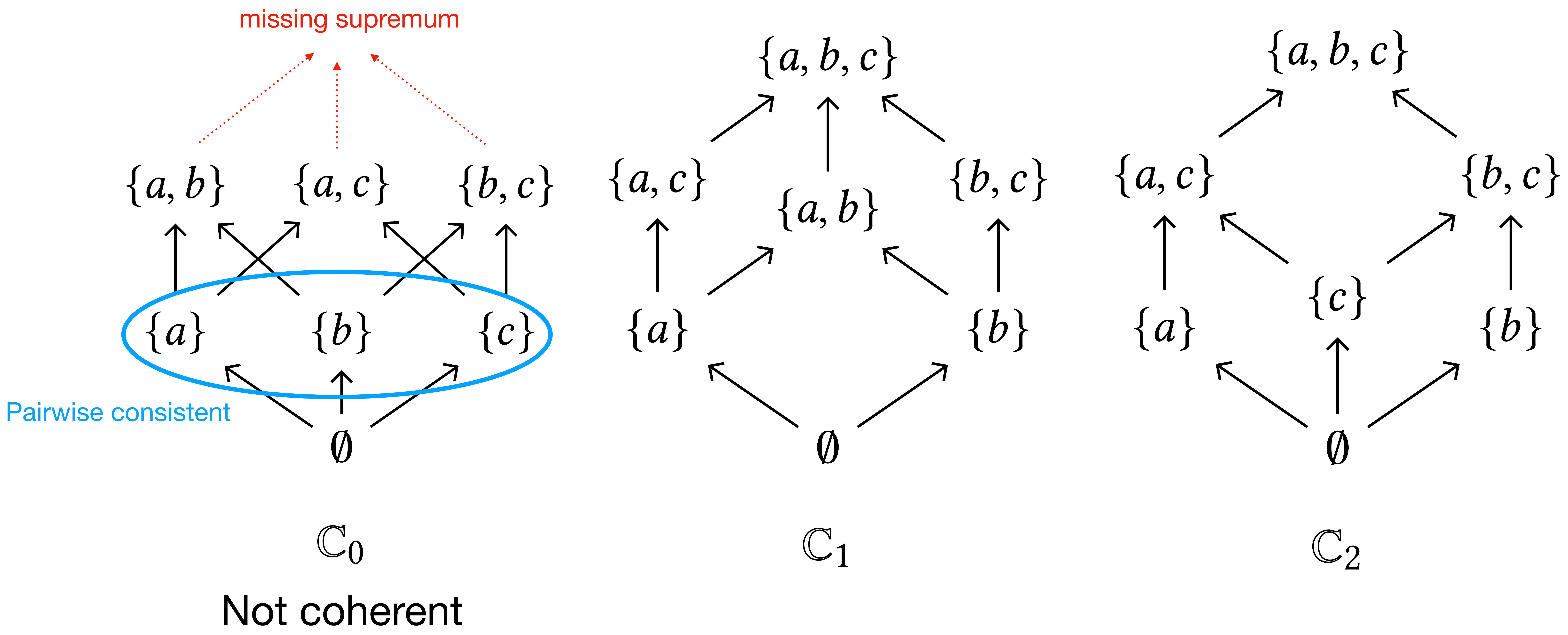


$\mathbb{C}_2$

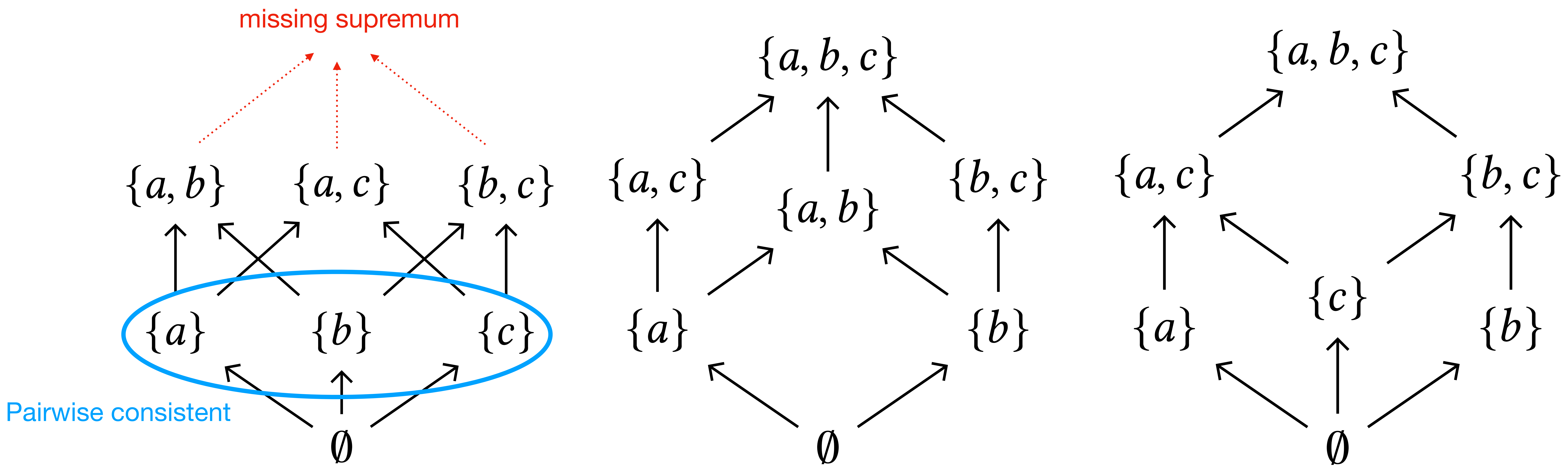
Stable configuration structures



Stable configuration structures



Stable configuration structures

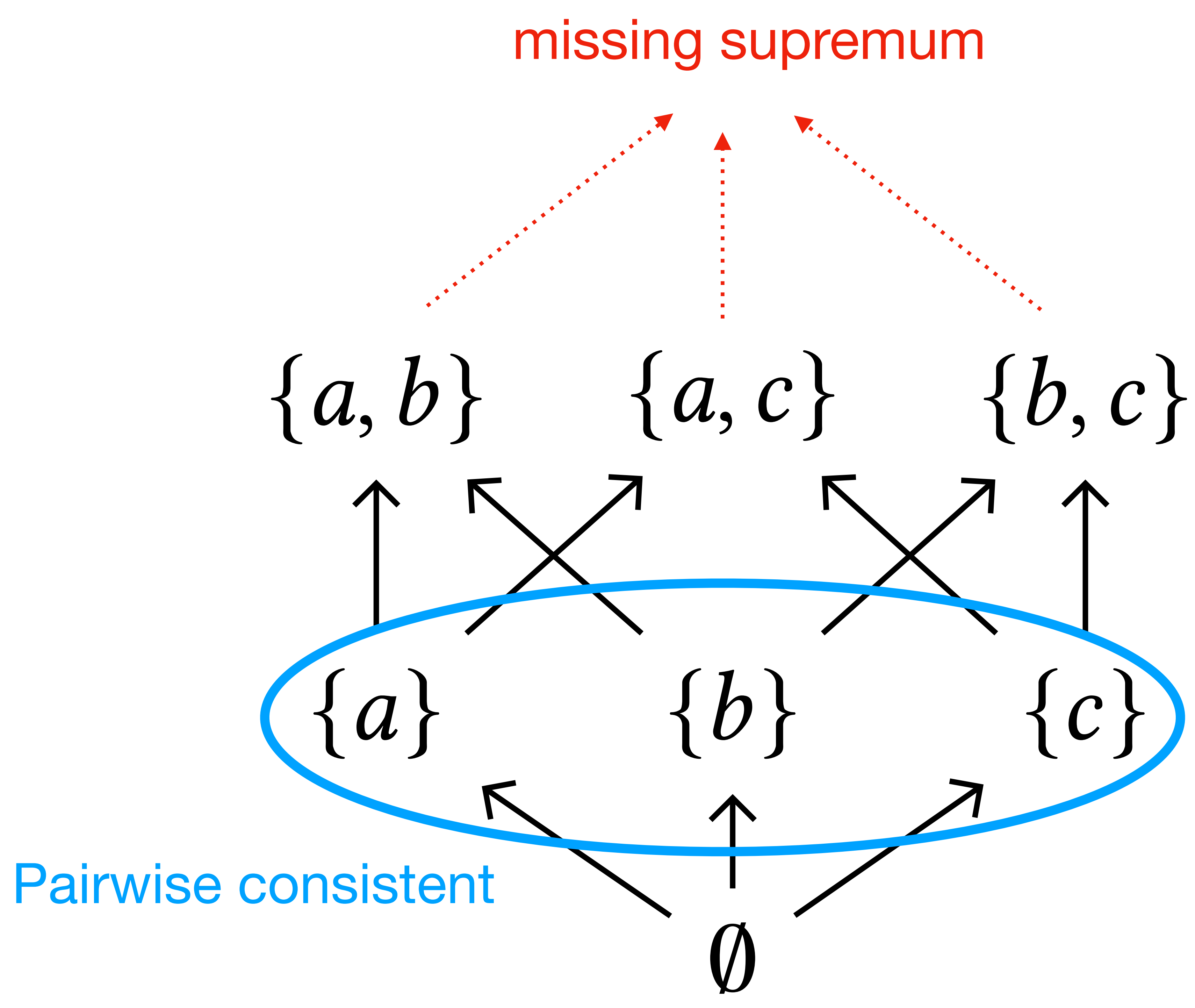


$\mathbb{C}_0$ $\mathbb{C}_1$ $\mathbb{C}_2$

Not coherent

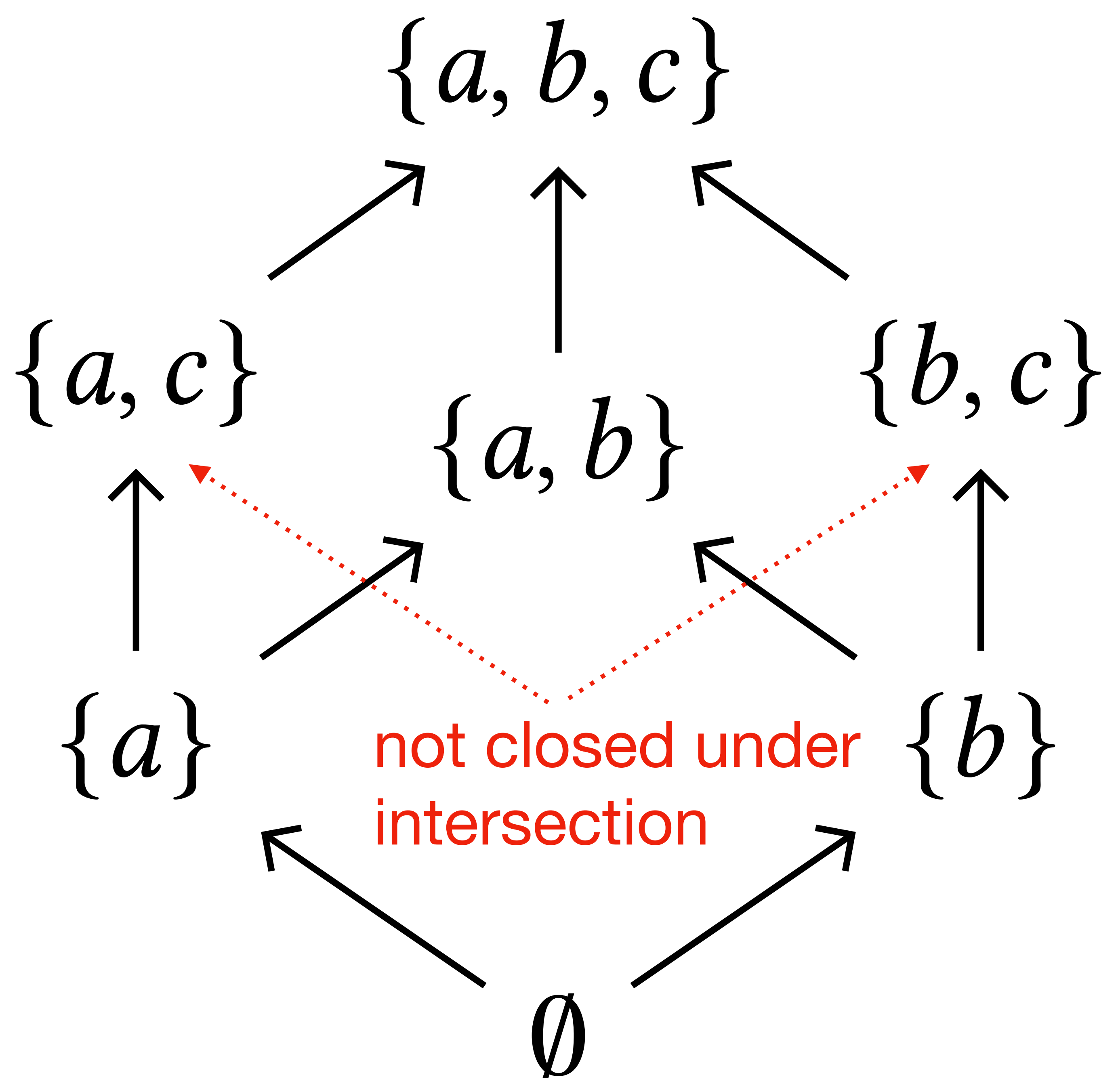
Not prime algebraic

Stable configuration structures

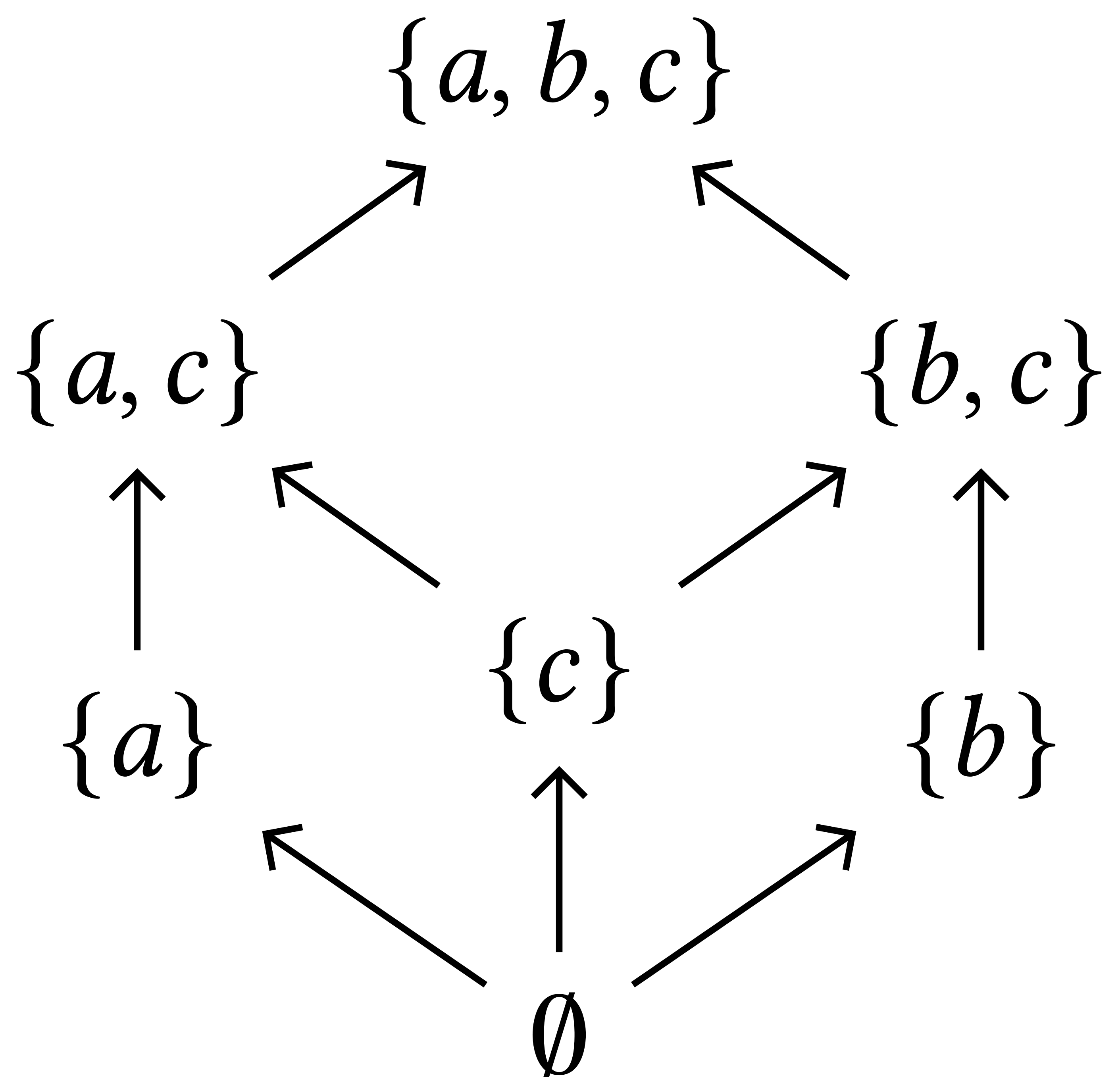


$\mathbb{C}_0$

Not coherent



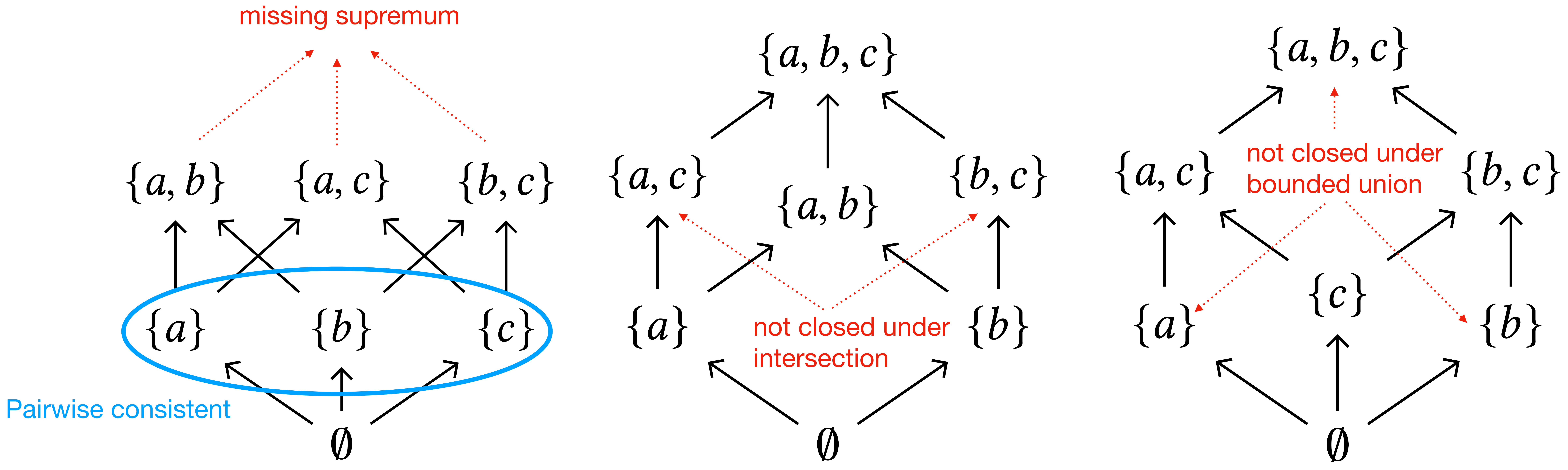
$\mathbb{C}_1$



$\mathbb{C}_2$

Not prime algebraic

Stable configuration structures



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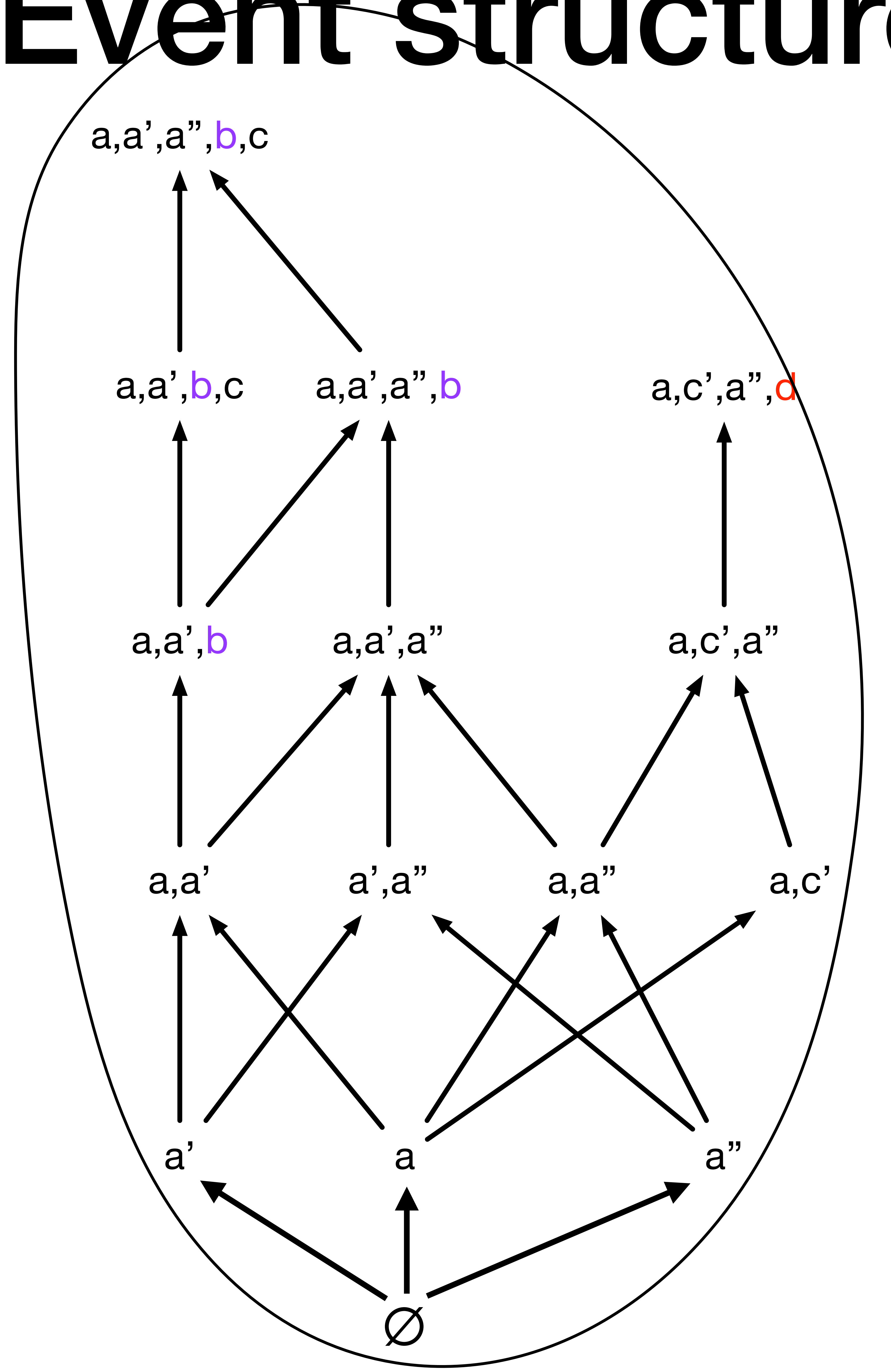
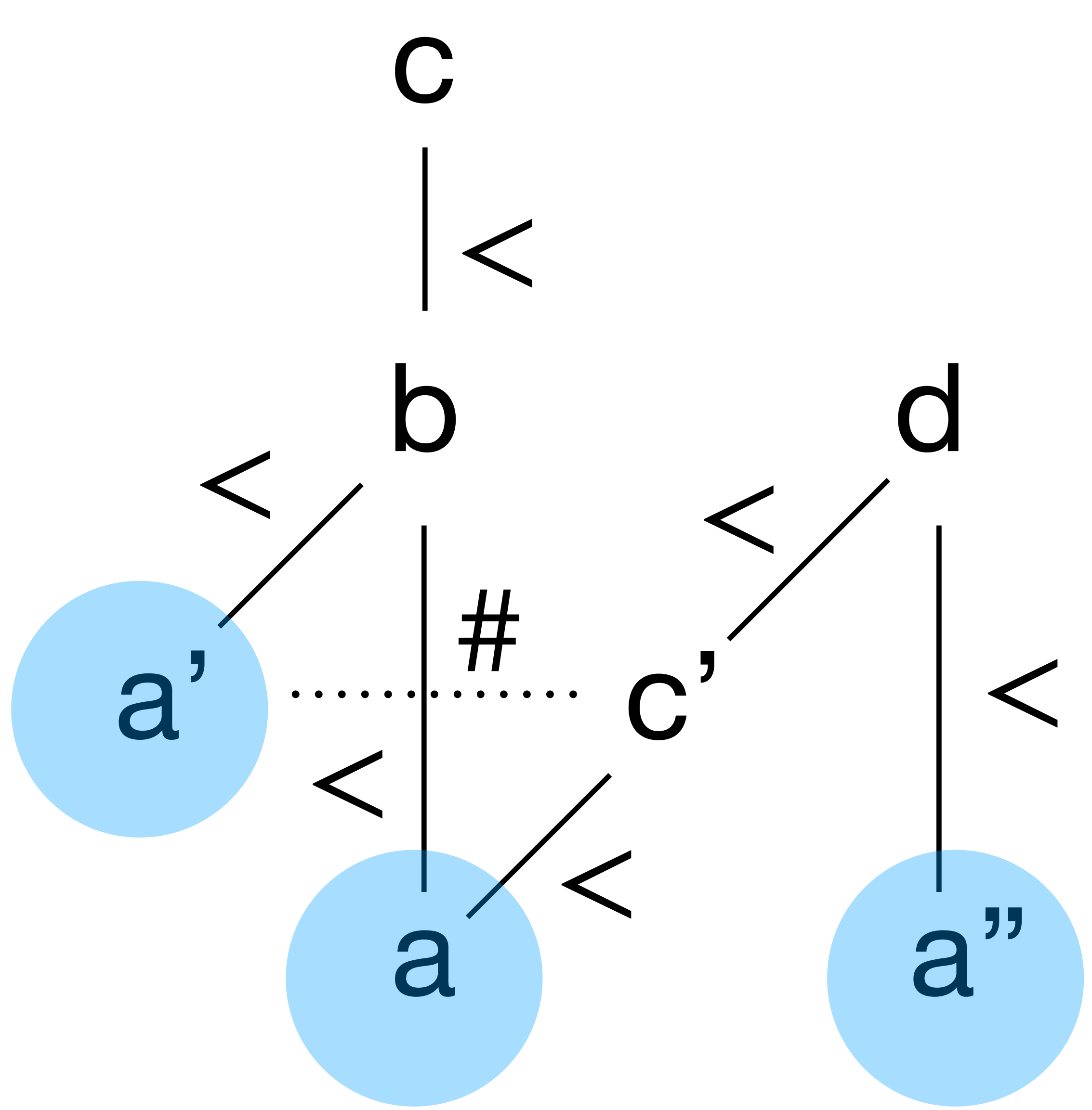
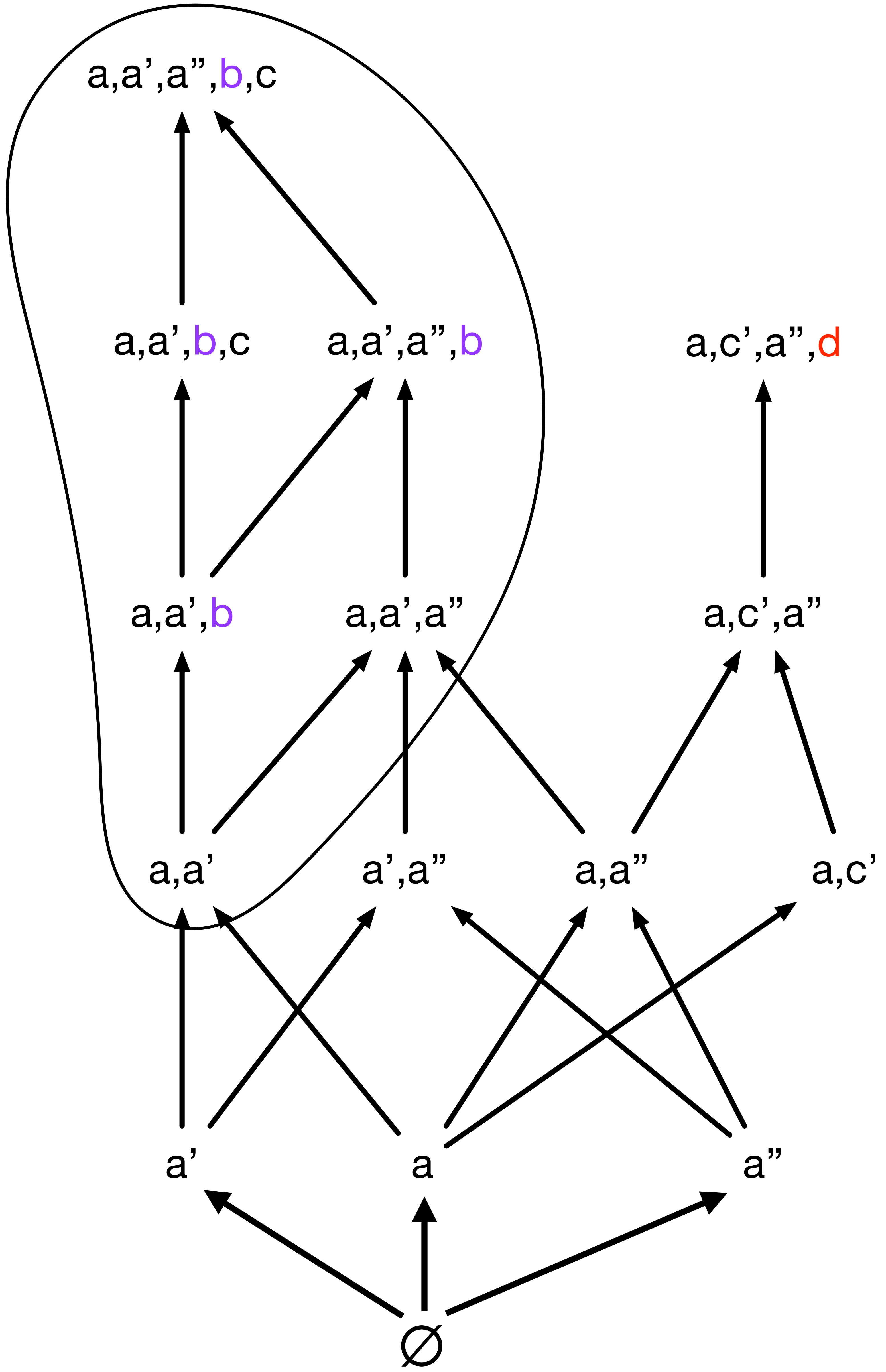
Event structure = local computations

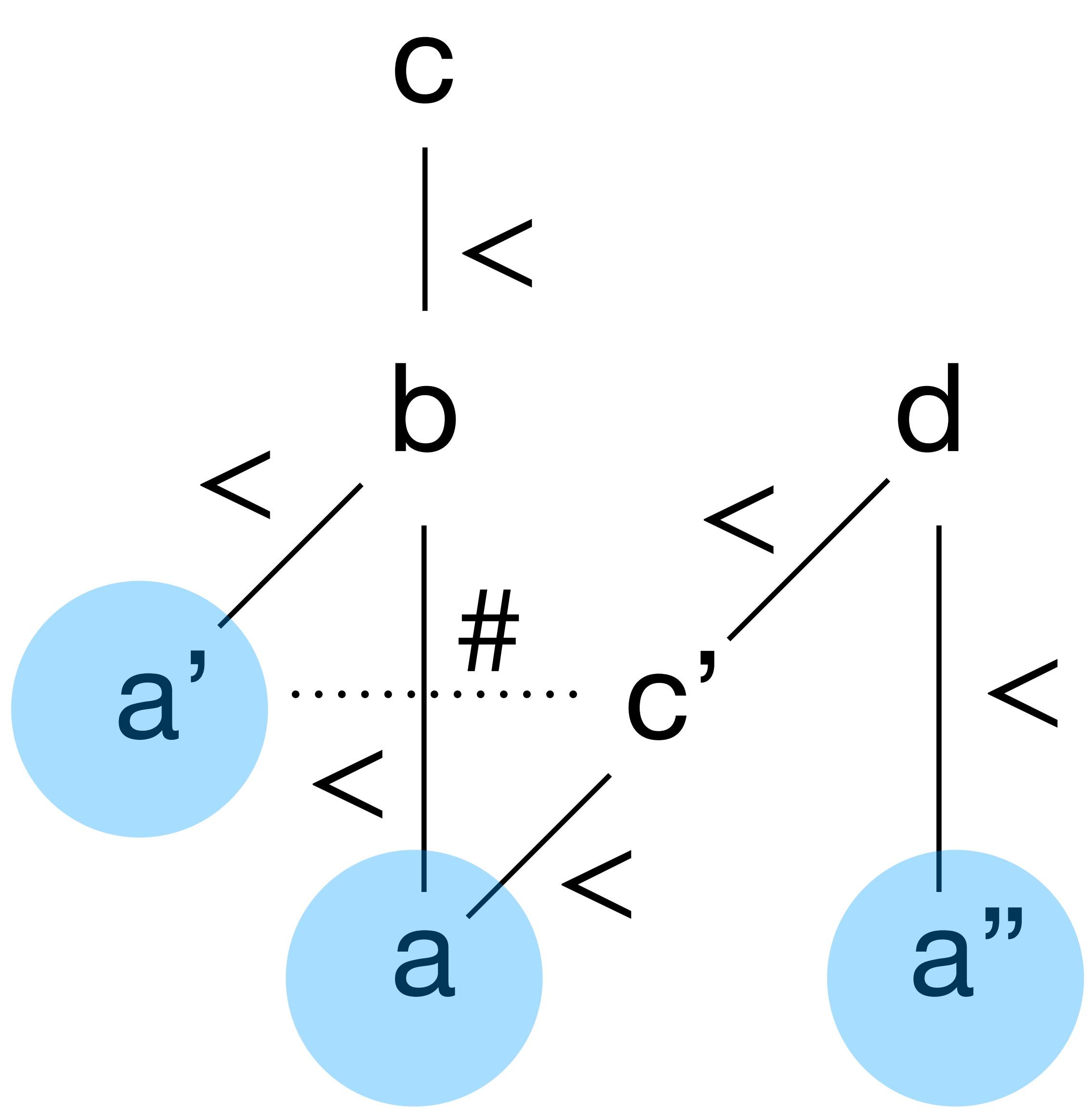




The enabled events are the minimal ones which are not in the forbidden set

Event structure = local computations





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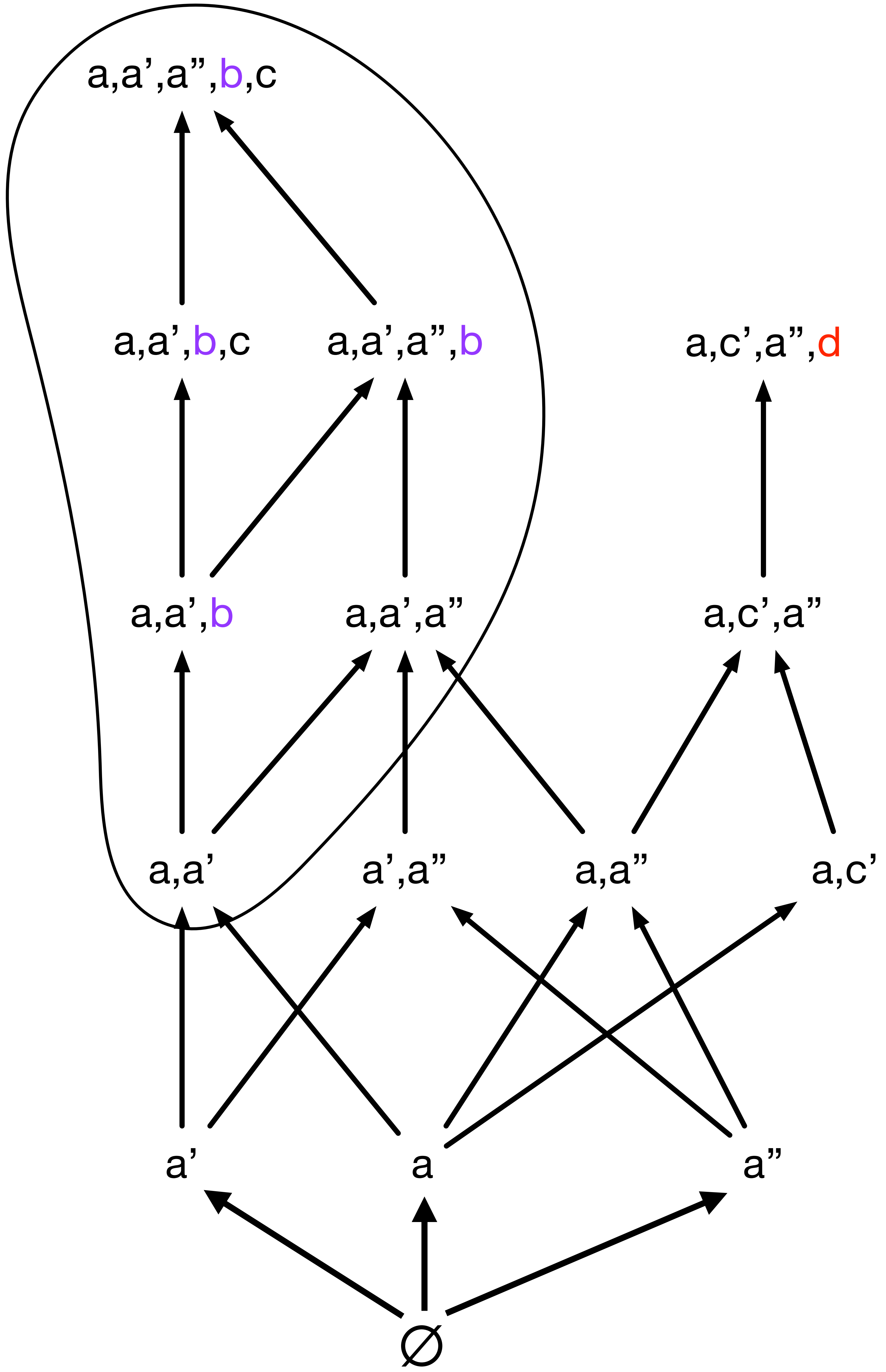

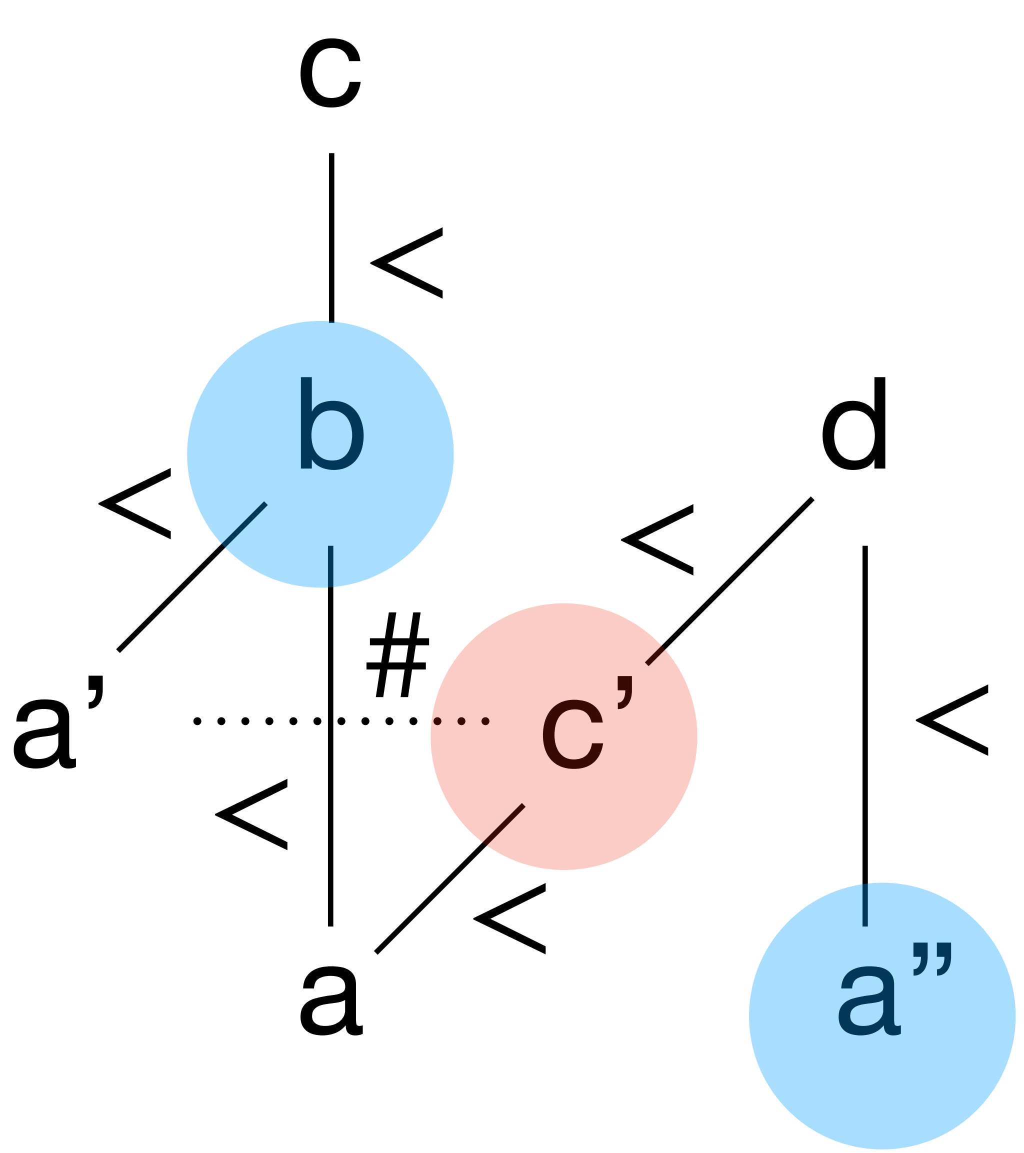

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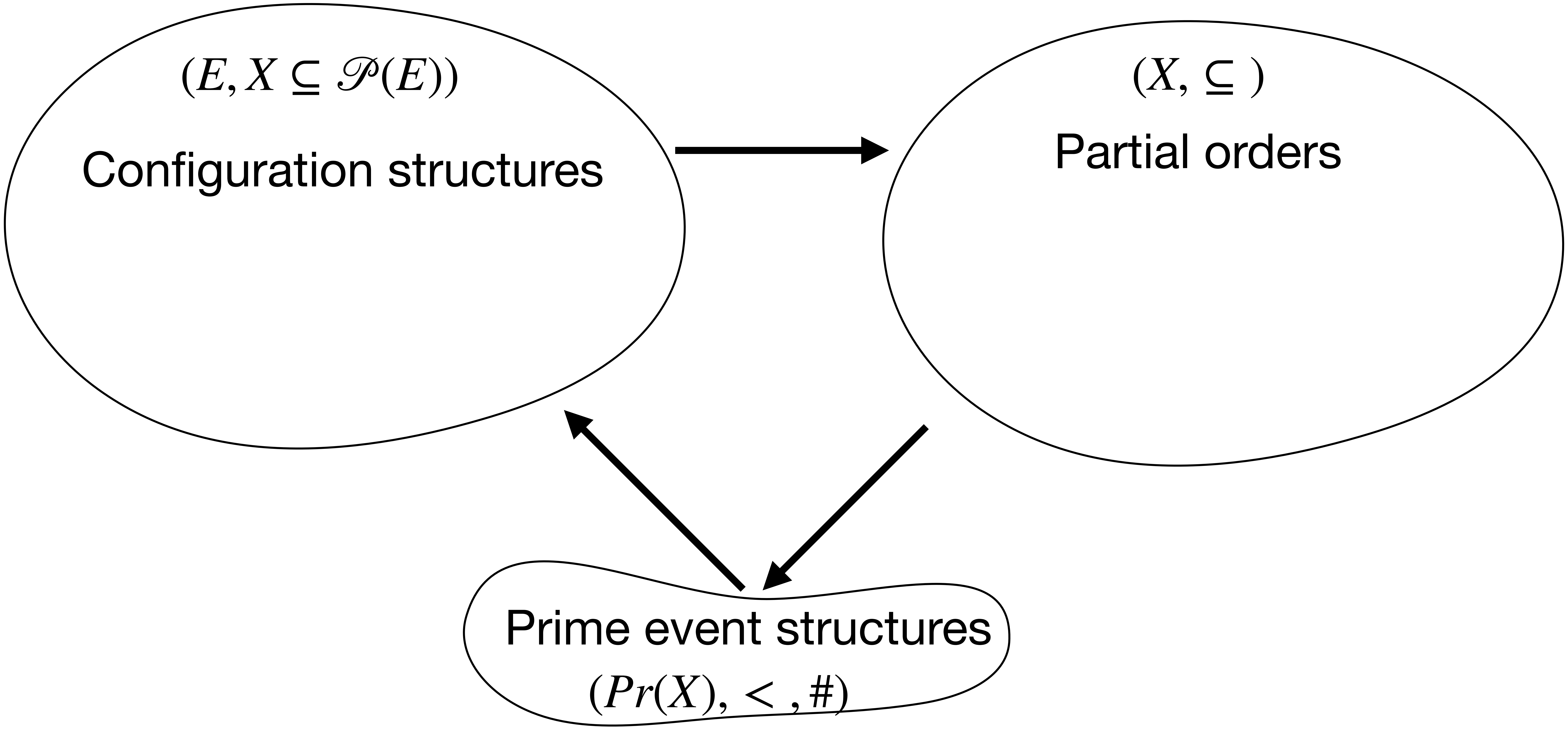
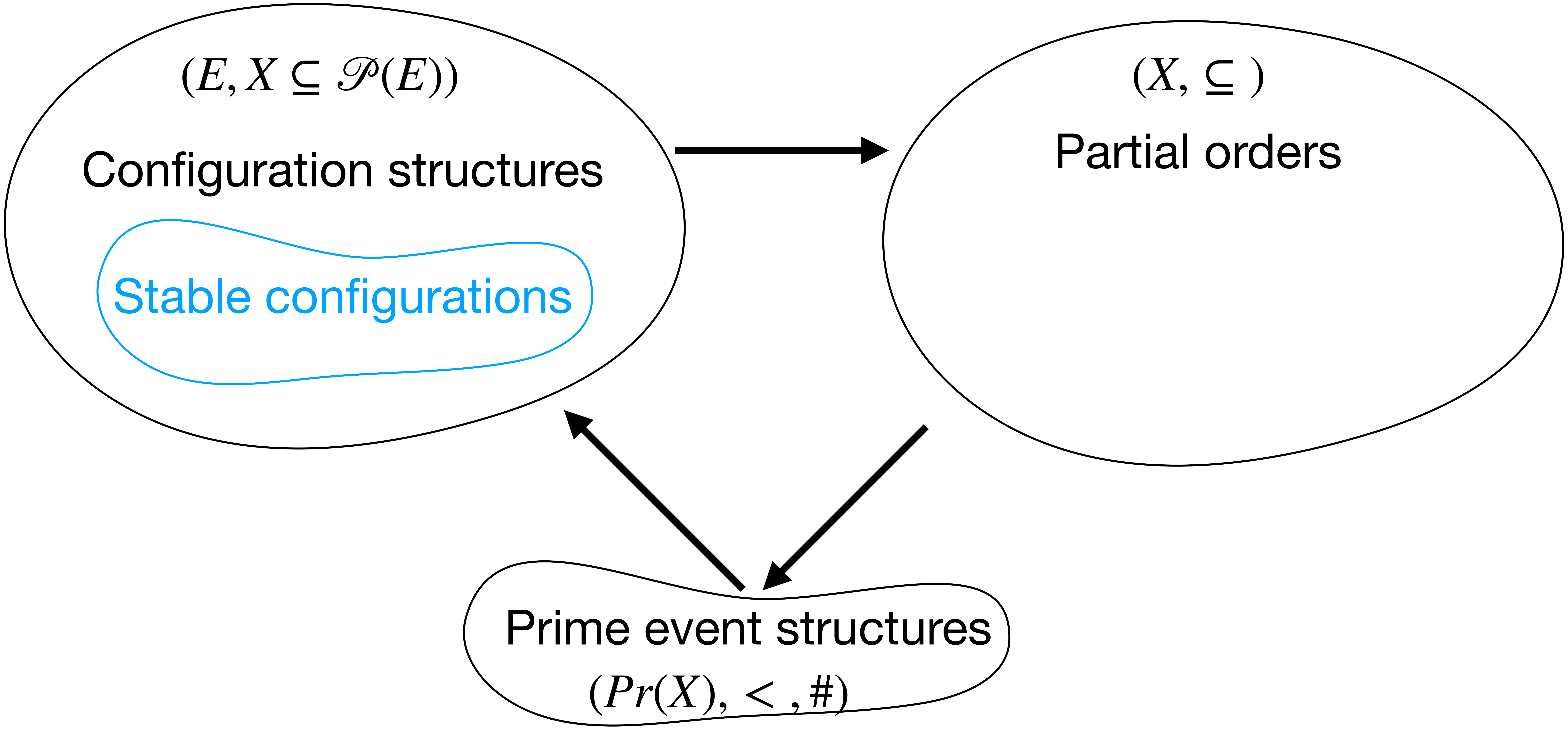
Residuation as computation



Residuation as computation



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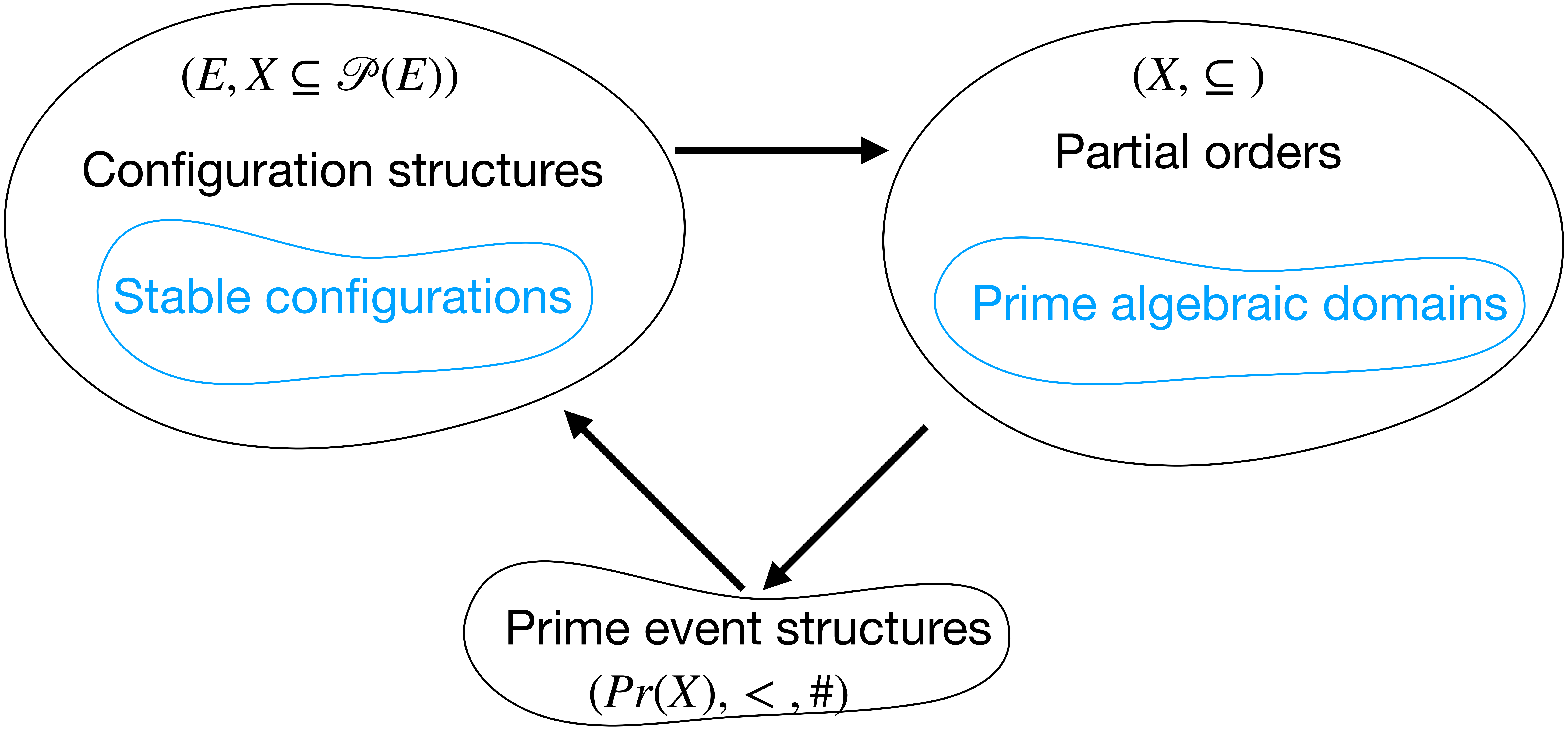

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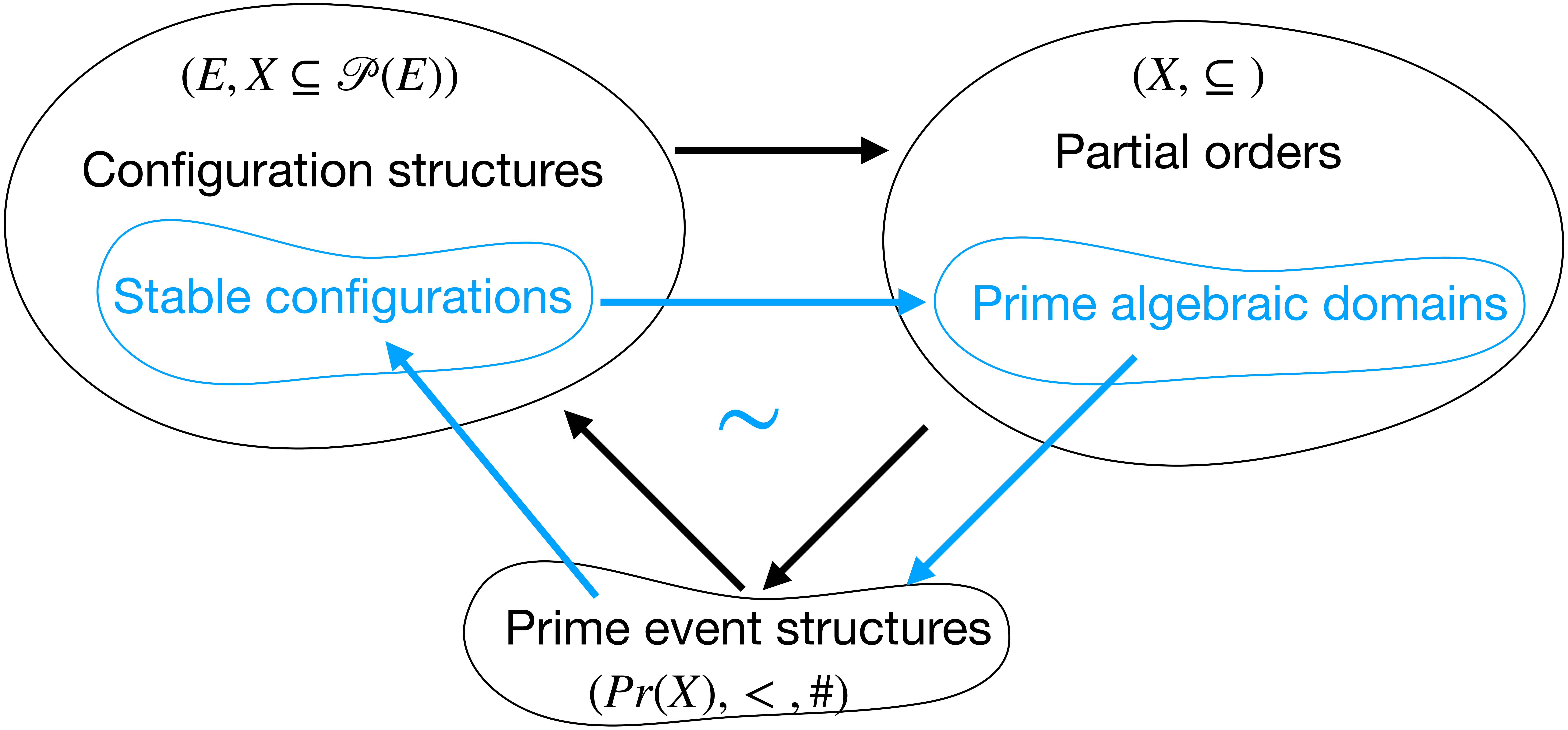

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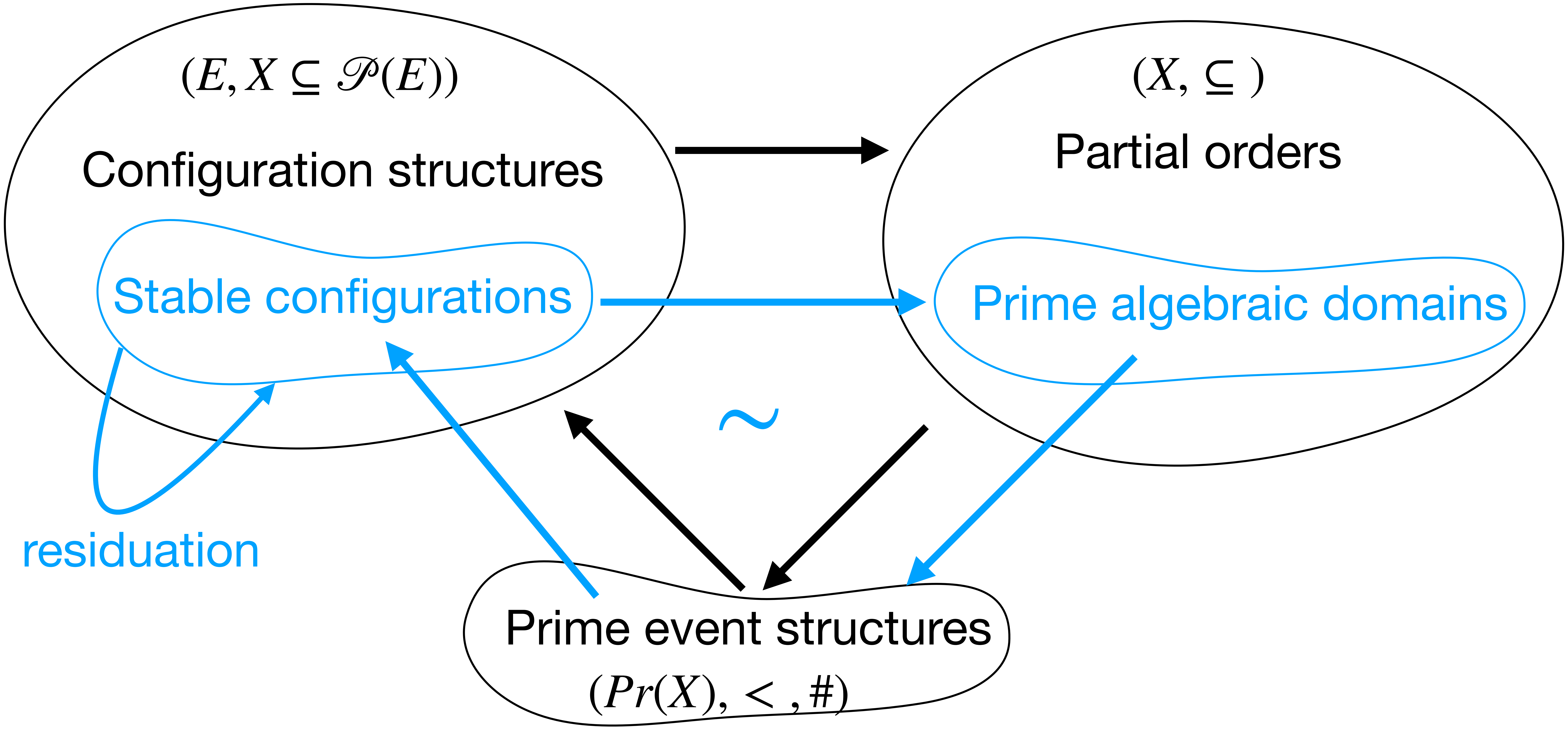

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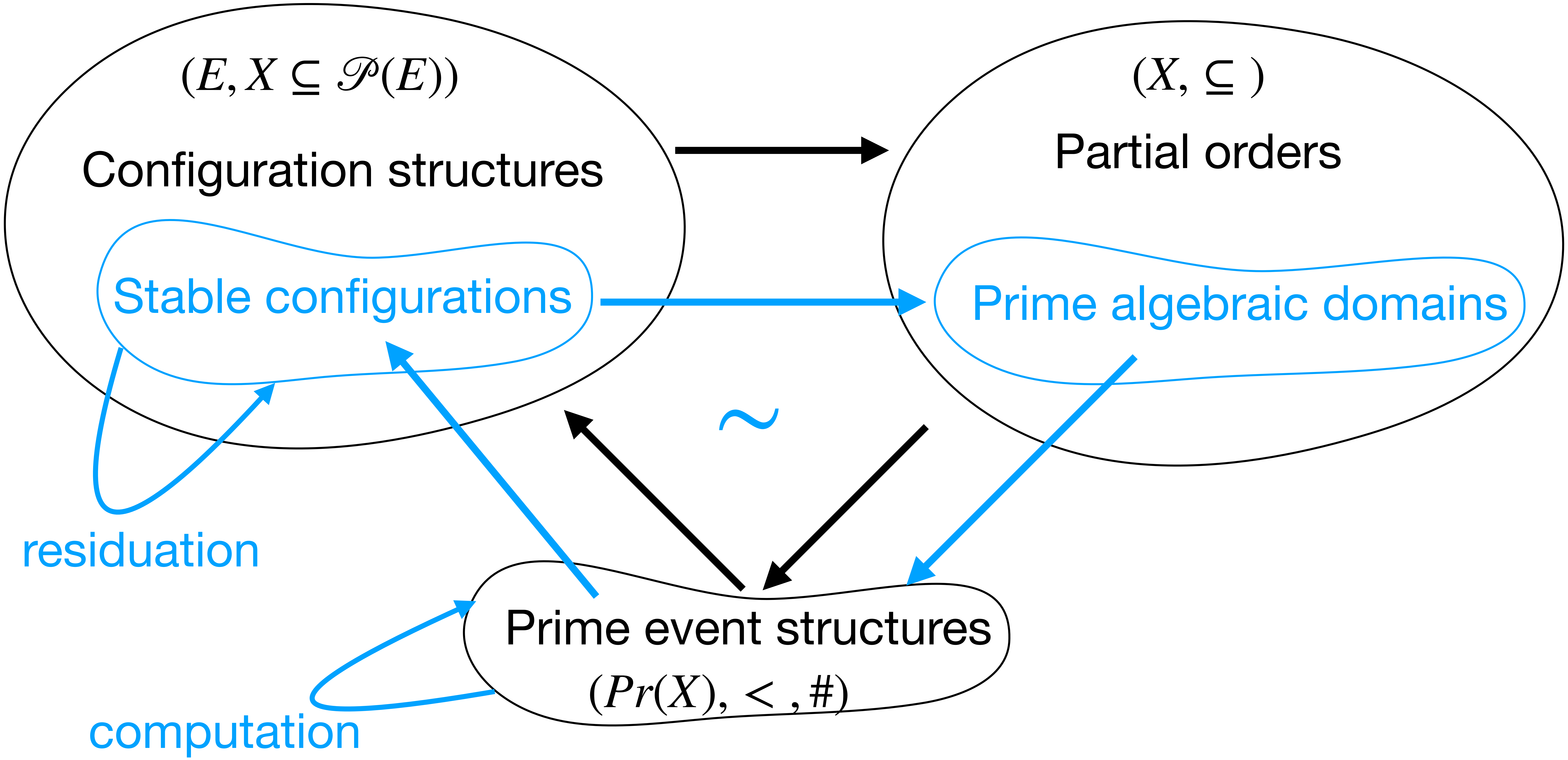

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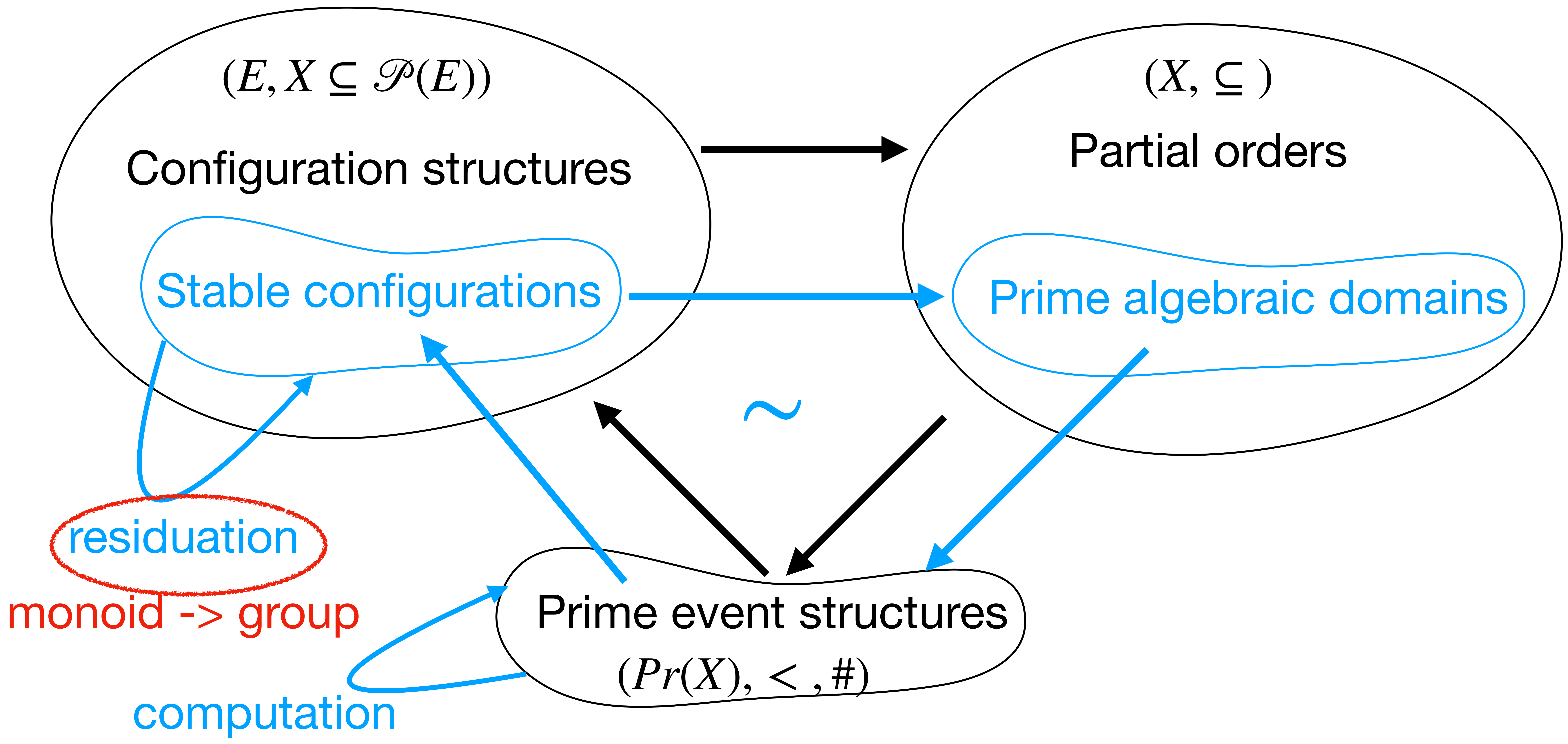
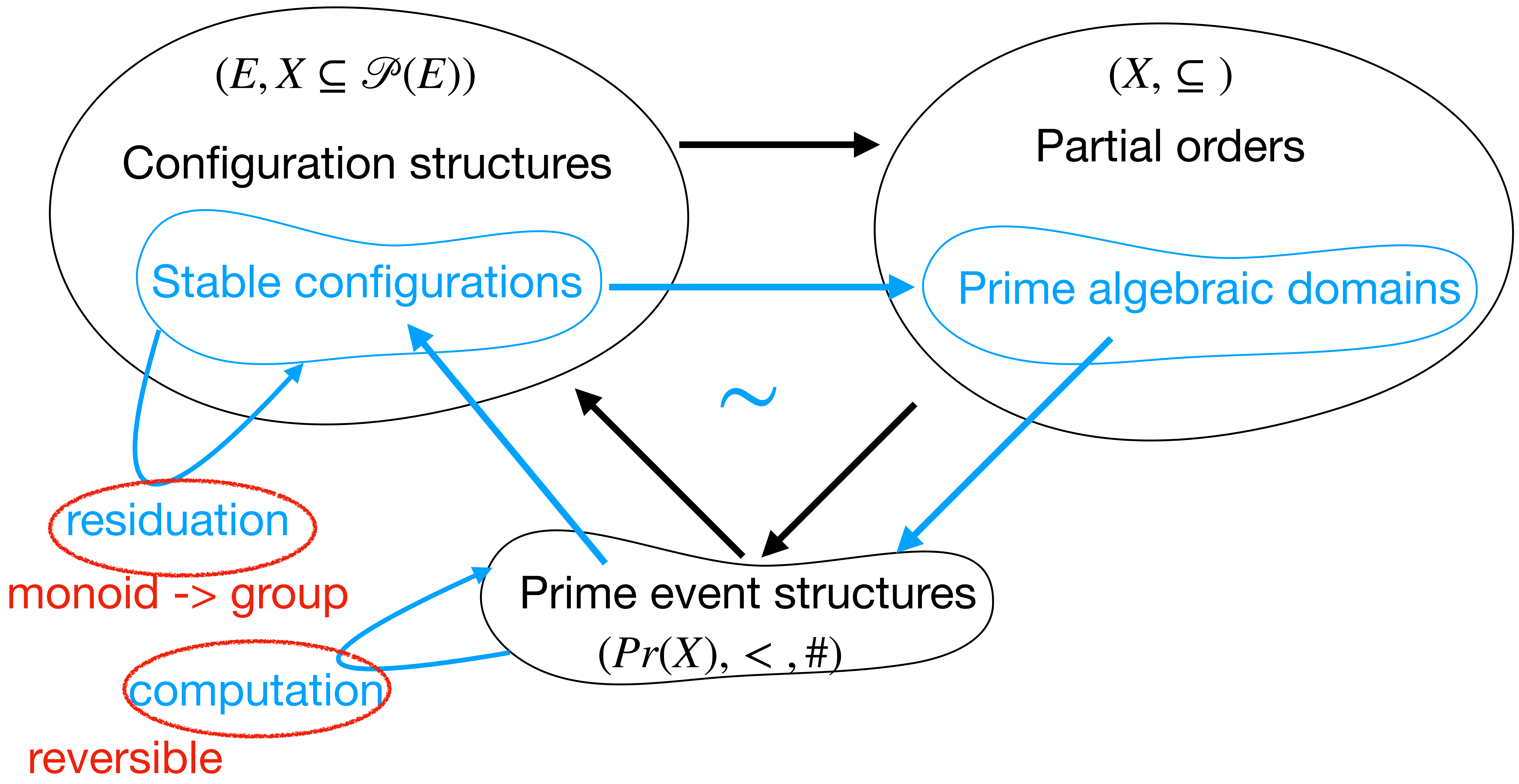


Residuation as computation



Symmetric residual

Definition 2.3 Let $\mathbb{C} \in \mathcal{C}_E$. For all finite $x \in \mathbb{C}$, we define the *residual* of $\mathbb{C}$ *after* x:

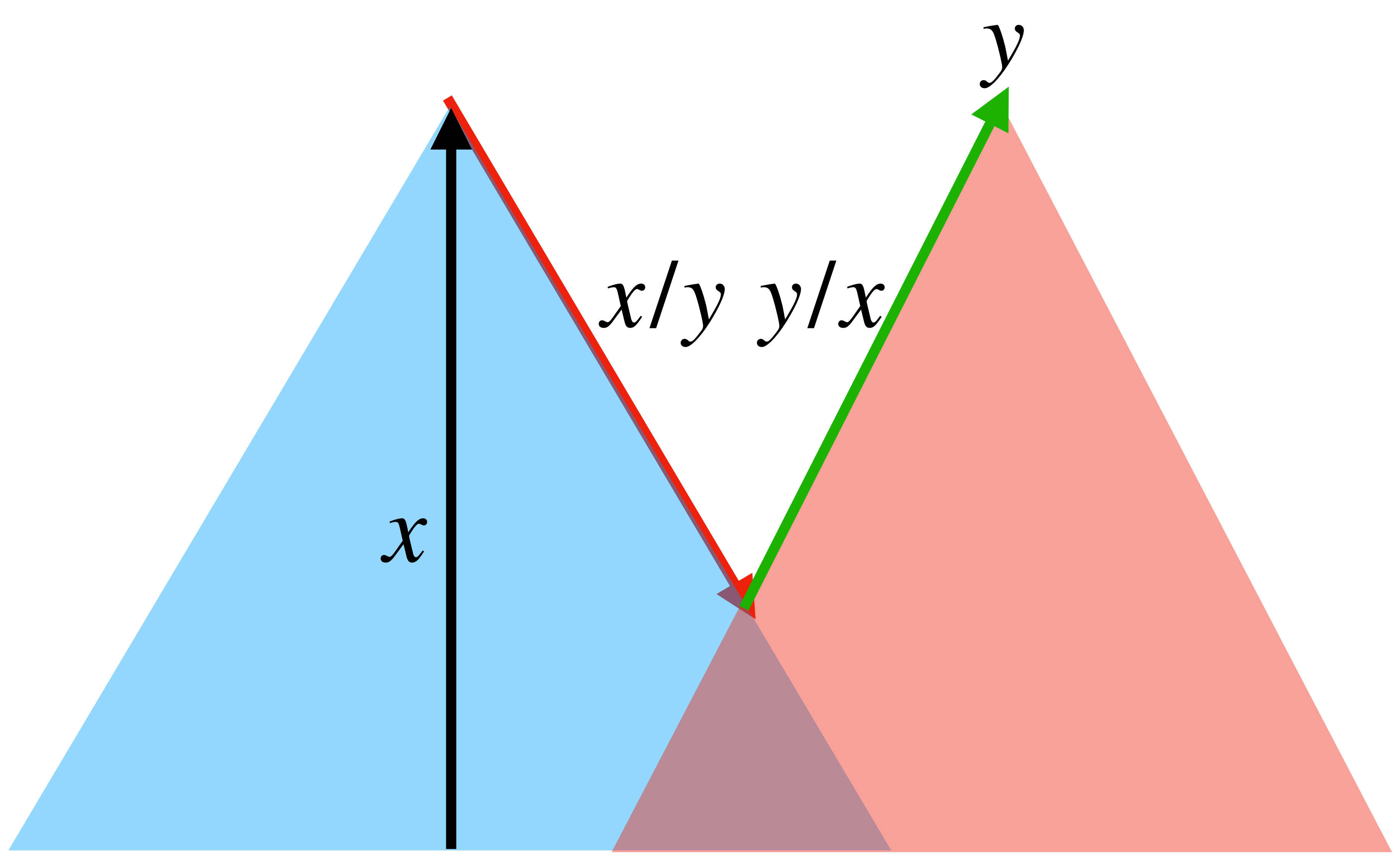
$$x \cdot \mathbb{C} := \langle E, \{z \in \mathcal{P}(E) \mid \exists y \in \uparrow^{\mathbb{C}}\{x\} : z = y \backslash x\} \rangle$$

where $y \backslash x := \{a \in y \mid a \notin x\}$ is the classical set difference.

$x \Delta y =_{def} x \backslash y \cup y \backslash x$

$(\mathscr{P}(E), \Delta)$ is a group

if $x \subseteq y$ then $x \Delta y = y \backslash x$



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$x \Delta y =_{def} x \backslash y \cup y \backslash x$ $\quad$ $(\mathscr{P}(E), \Delta)$ is a group $\quad$ if $x \subseteq y$ then $x \Delta y = y \backslash x$

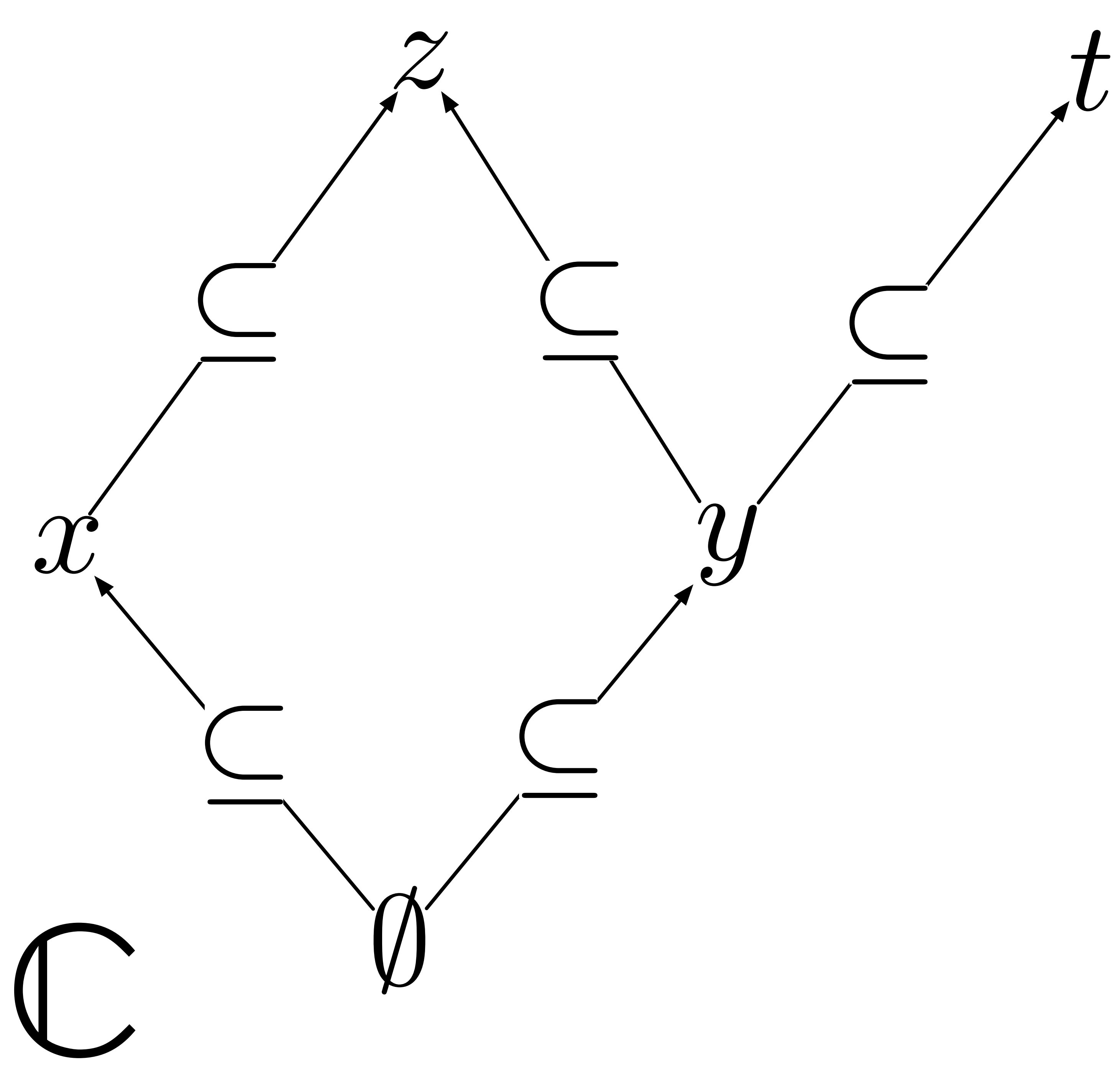
Definition 3.1 Let $\mathbb{C} \in \mathcal{C}_E$. For all finite $x \in \mathbb{C}$, we define the *symmetric residual of* $\mathbb{C}$ *after* x:

$$x \odot \mathbb{C} := \langle E, \{z \in \mathcal{P}(E) \mid \exists y \in \mathbb{C} : z = y \vartriangle x\}\rangle.$$

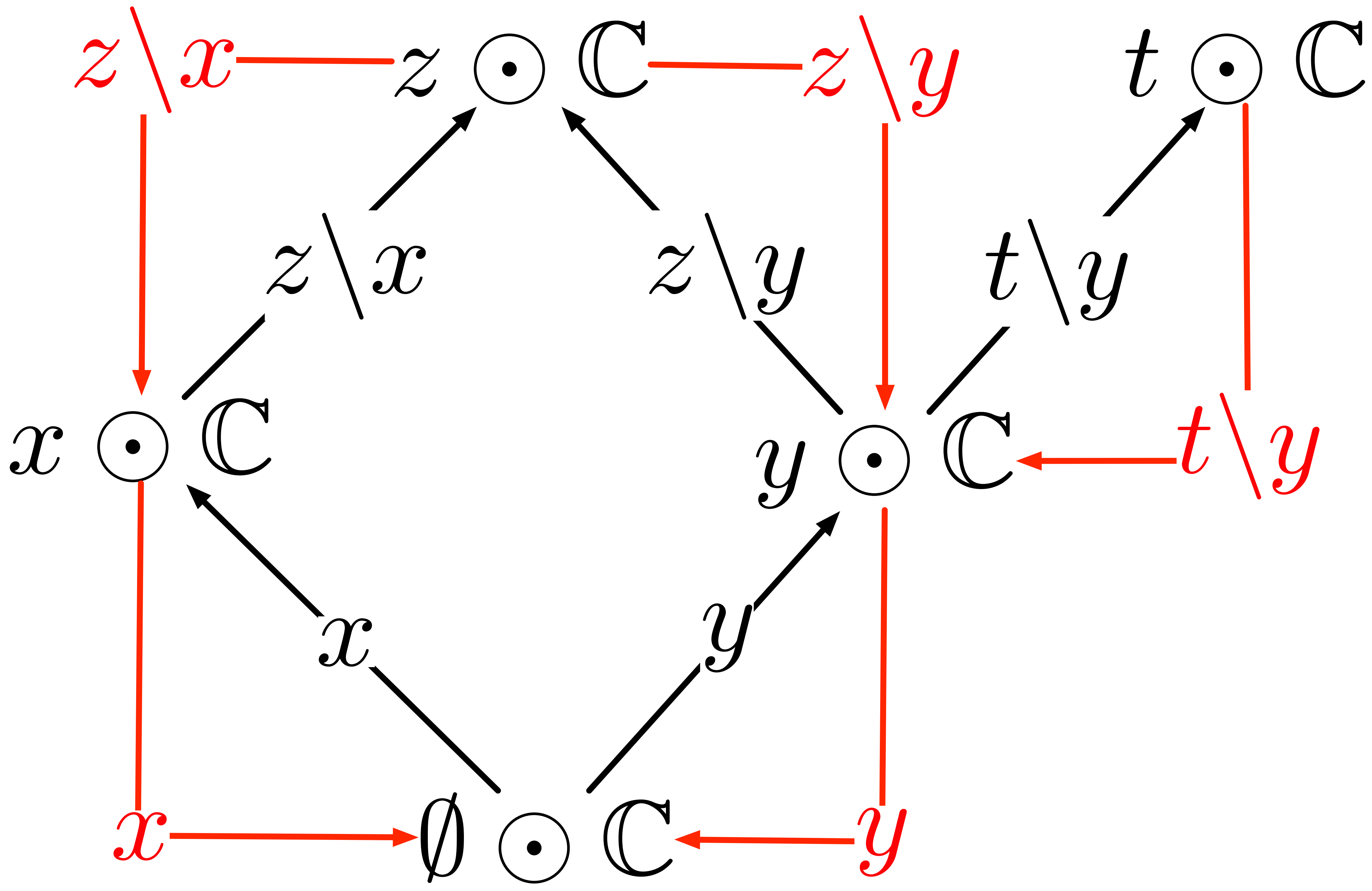
Proposition 3.2 (Group action) *The operator* $(\odot) : \mathcal{P}_{fin}(E) \times \mathcal{C}_E \to \mathcal{C}_E$ *is a* group action *on configuration structures, i.e.:*

- *for all finite configurations* x, y, *if* $x \in \mathbb{C}$ *and* $y \in x \odot \mathbb{C}$, *then* $x \odot (y \odot \mathbb{C}) = (x \vartriangle y) \odot \mathbb{C}$.
- $\emptyset \odot \mathbb{C} = \mathbb{C}$.

Reversible computation

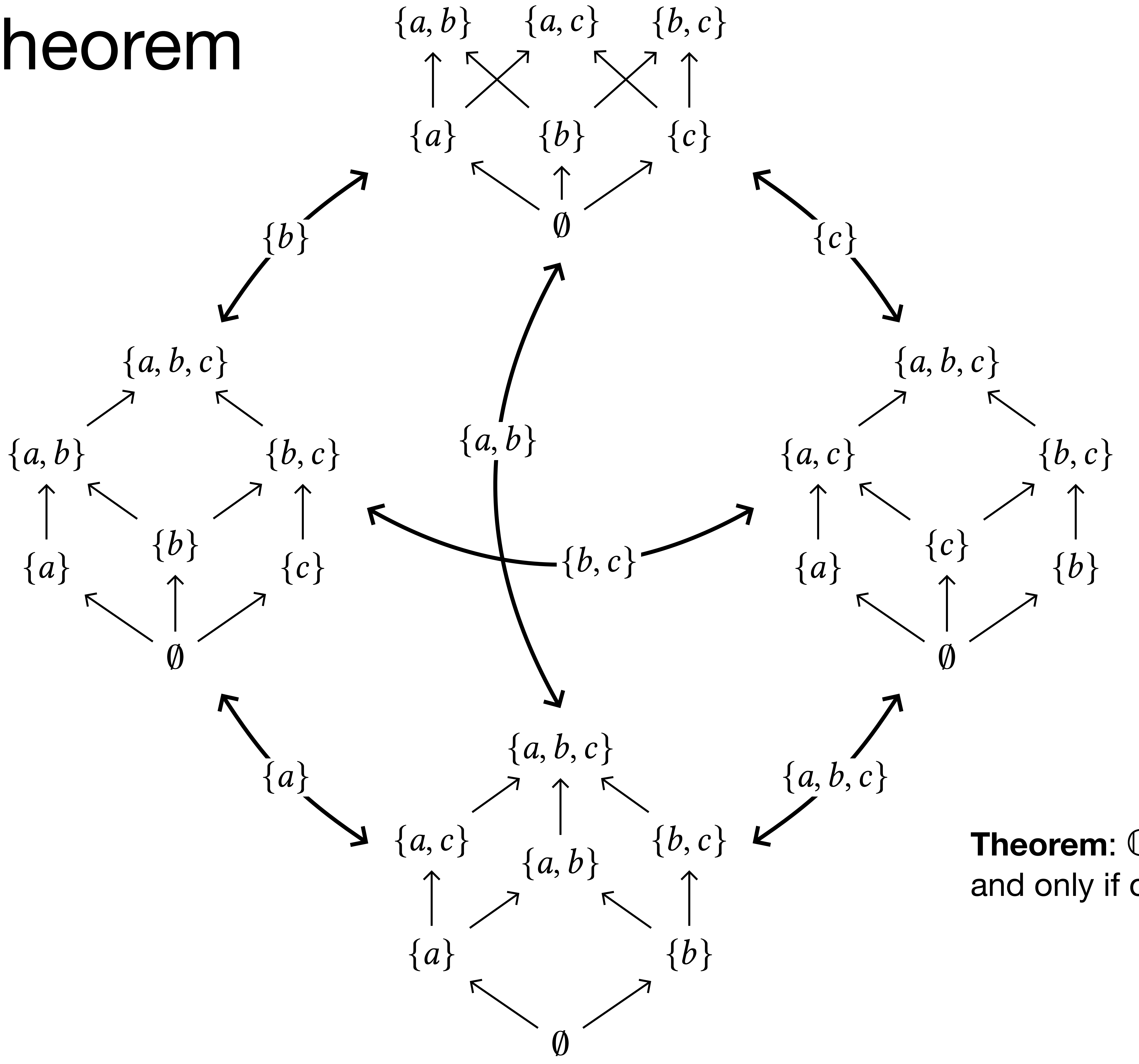


+ symmetric residuation



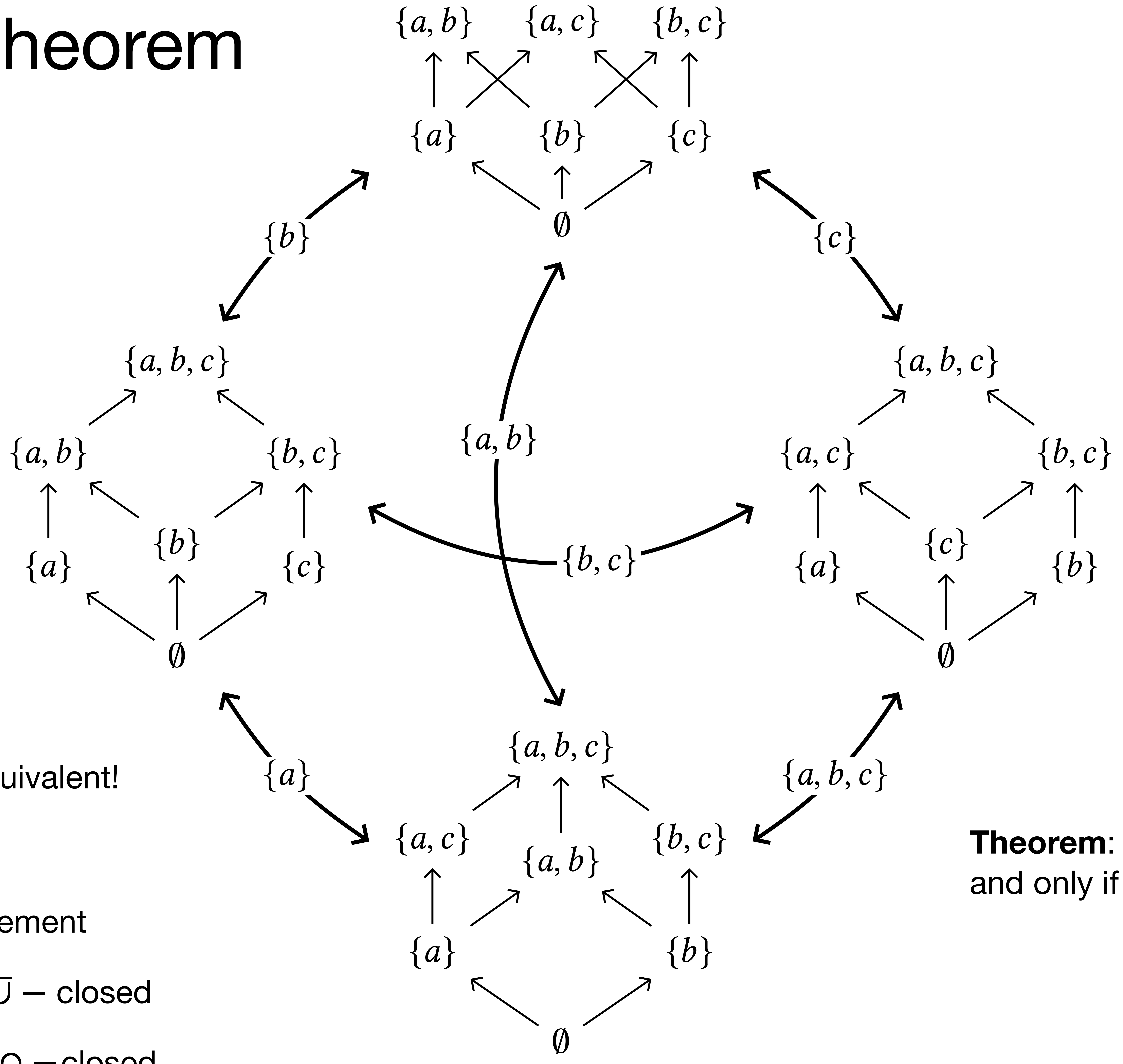
(conservative) labelled transition system as a group action's orbit

Stable orbit theorem



Theorem: $\mathbb{C}$ is a prime event structure if and only if one the point in its orbit is.

Stable orbit theorem

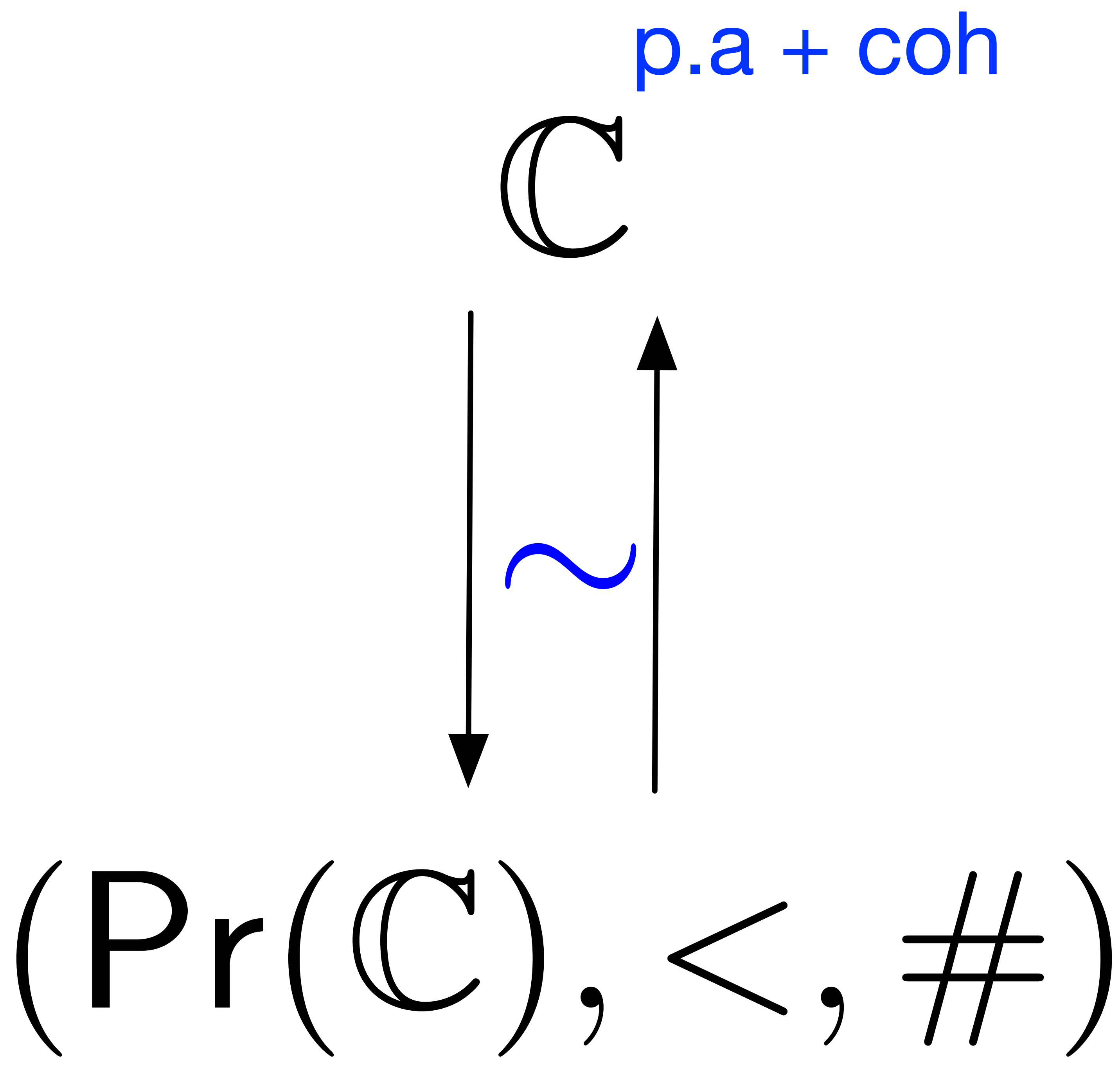


Property: The following are equivalent!

- $\mathbb{C}$ is not an event structure
- orb($\mathbb{C}$) has an incoherent element
- orb($\mathbb{C}$) has an element not $\overline{\cup}-$closed
- orb($\mathbb{C}$) has an element not $\cap-$closed

Theorem: $\mathbb{C}$ is a prime event structure if and only if one the point in its orbit is.

Interpreting symmetric residual

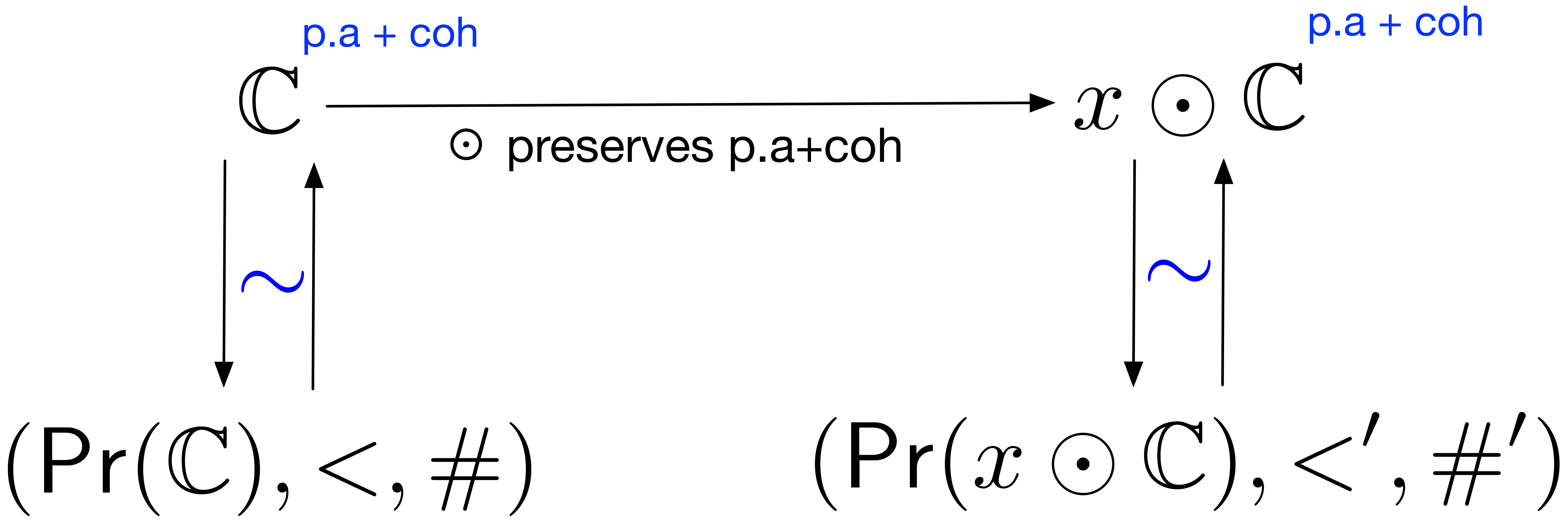


Interpreting symmetric residual

$$\mathbb{C}^{\text{p.a + coh}} \;\underset{\sim}{\rightleftarrows}\; (\mathrm{Pr}(\mathbb{C}), <, \#)$$

$\odot$ preserves p.a+coh

Interpreting symmetric residual



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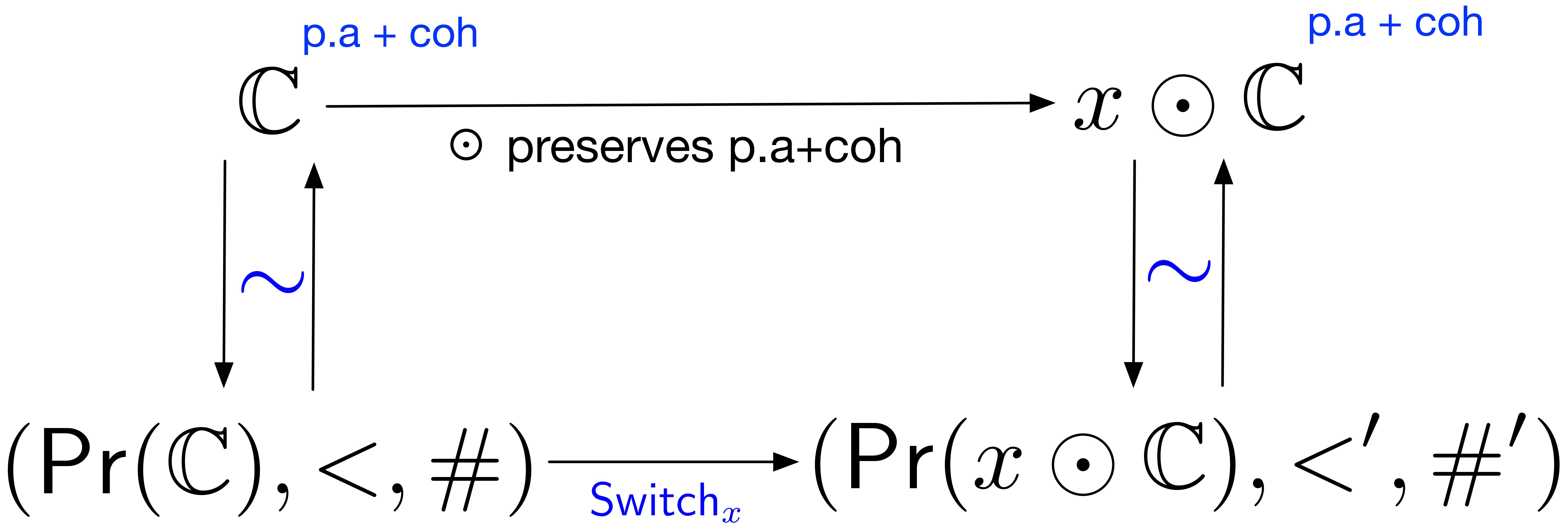
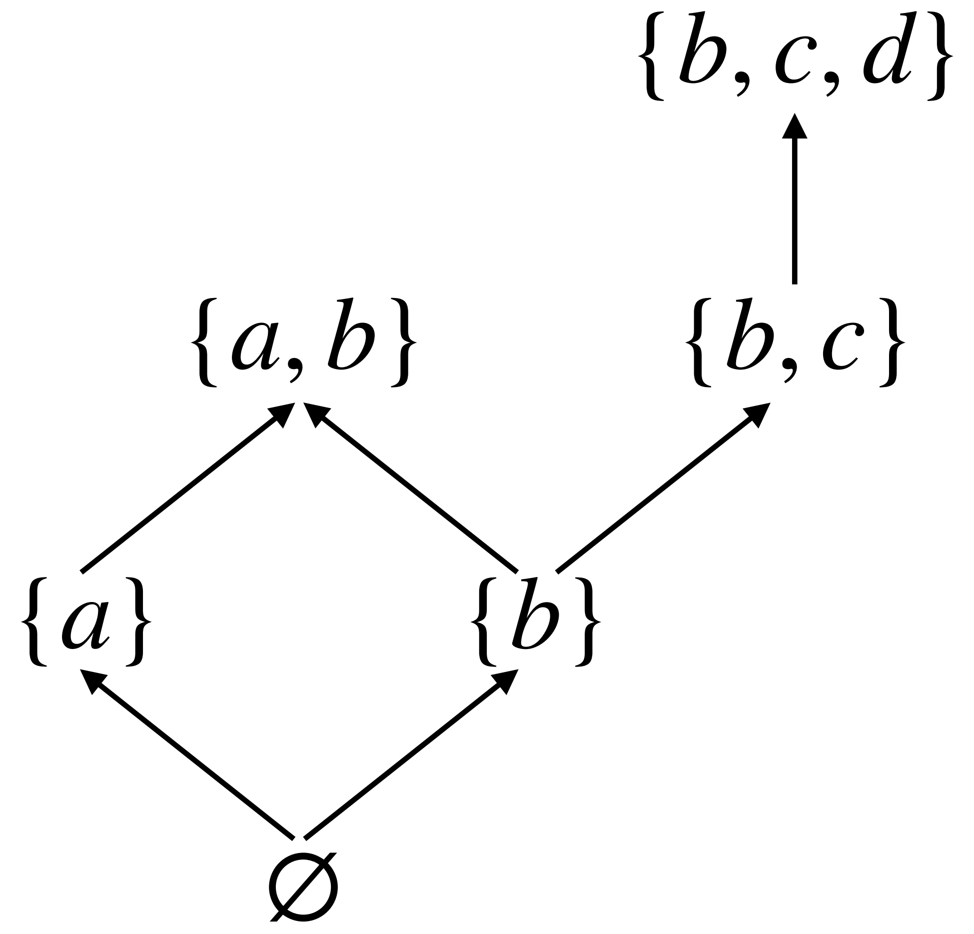


Event structure switch



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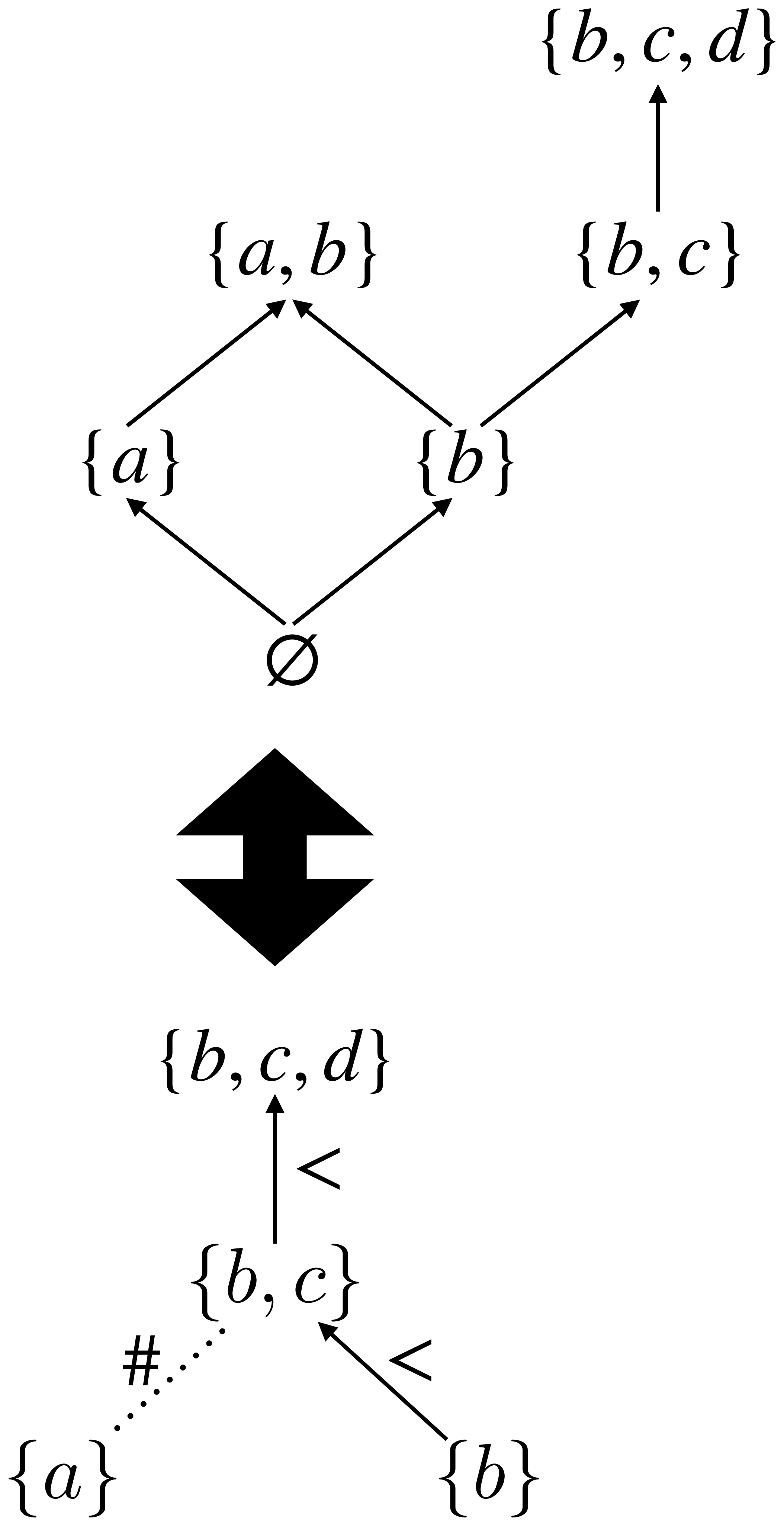

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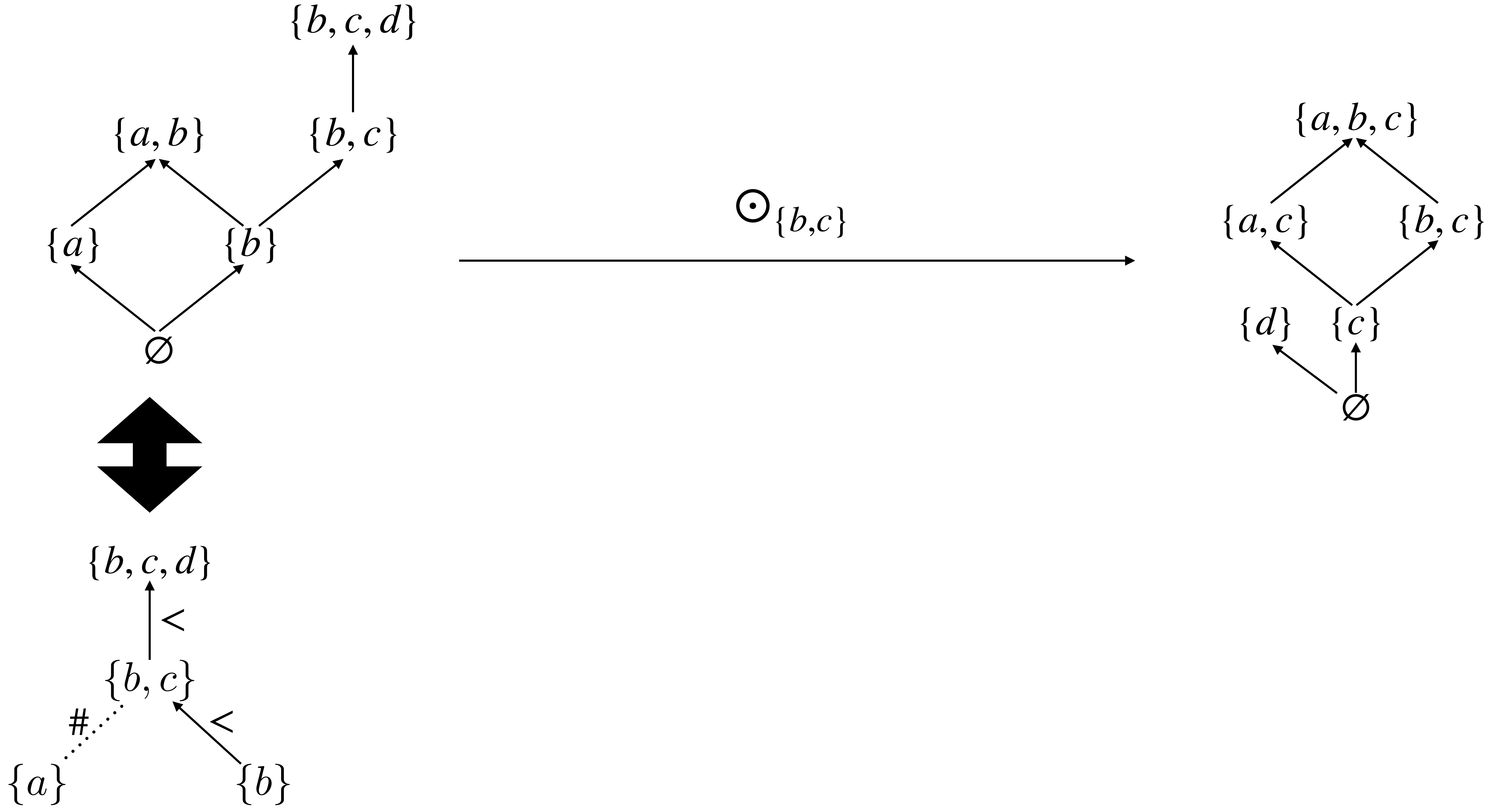

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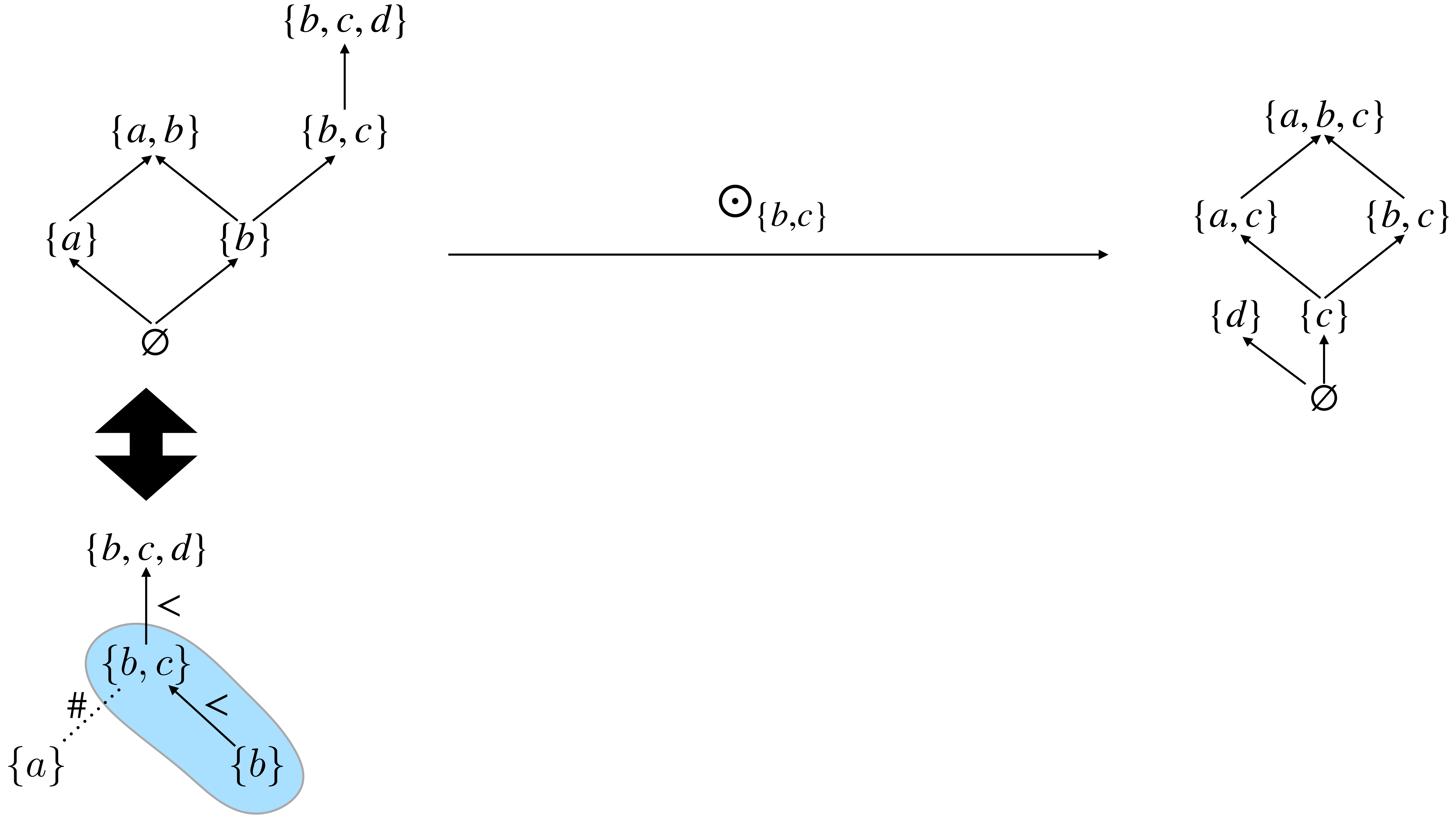

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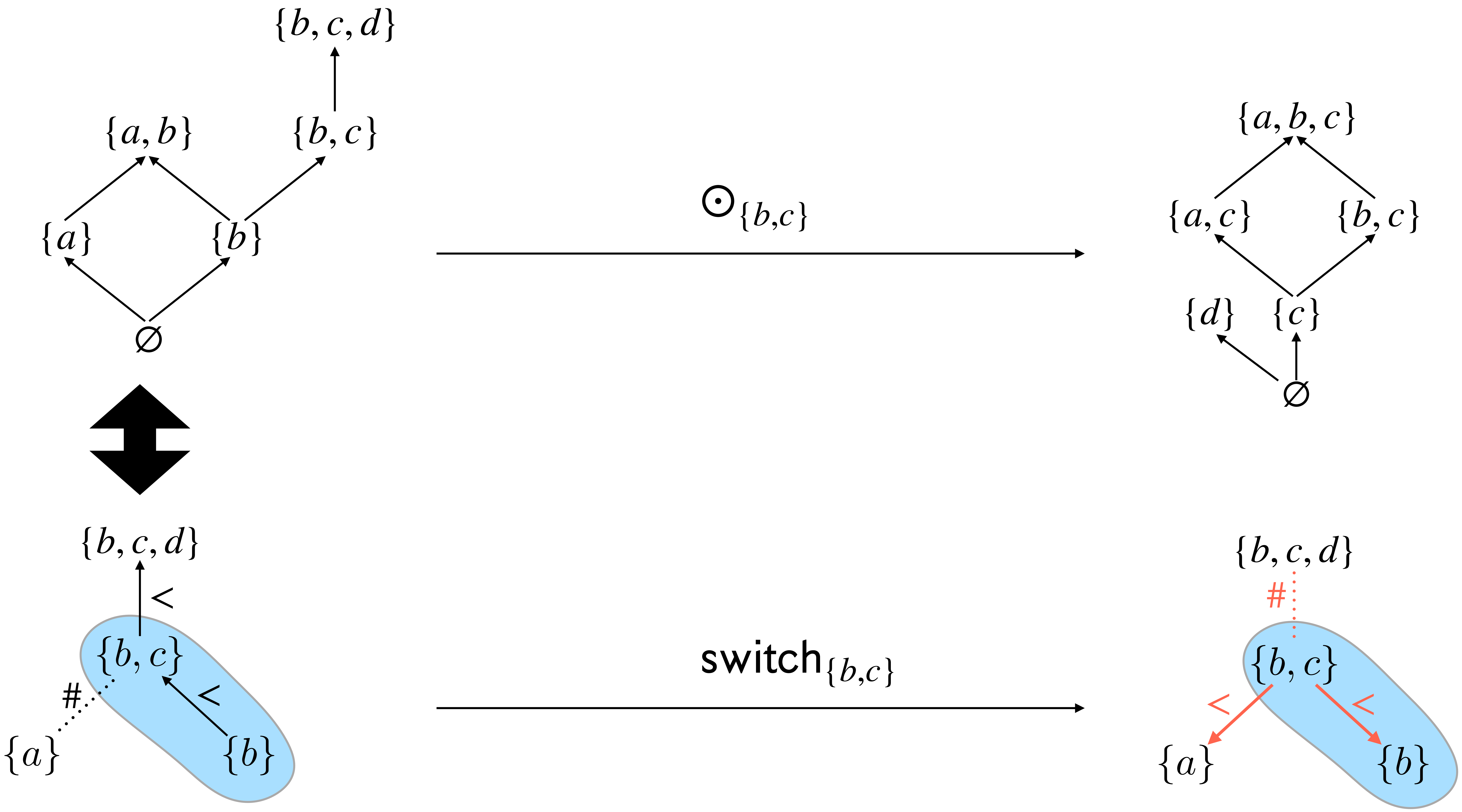

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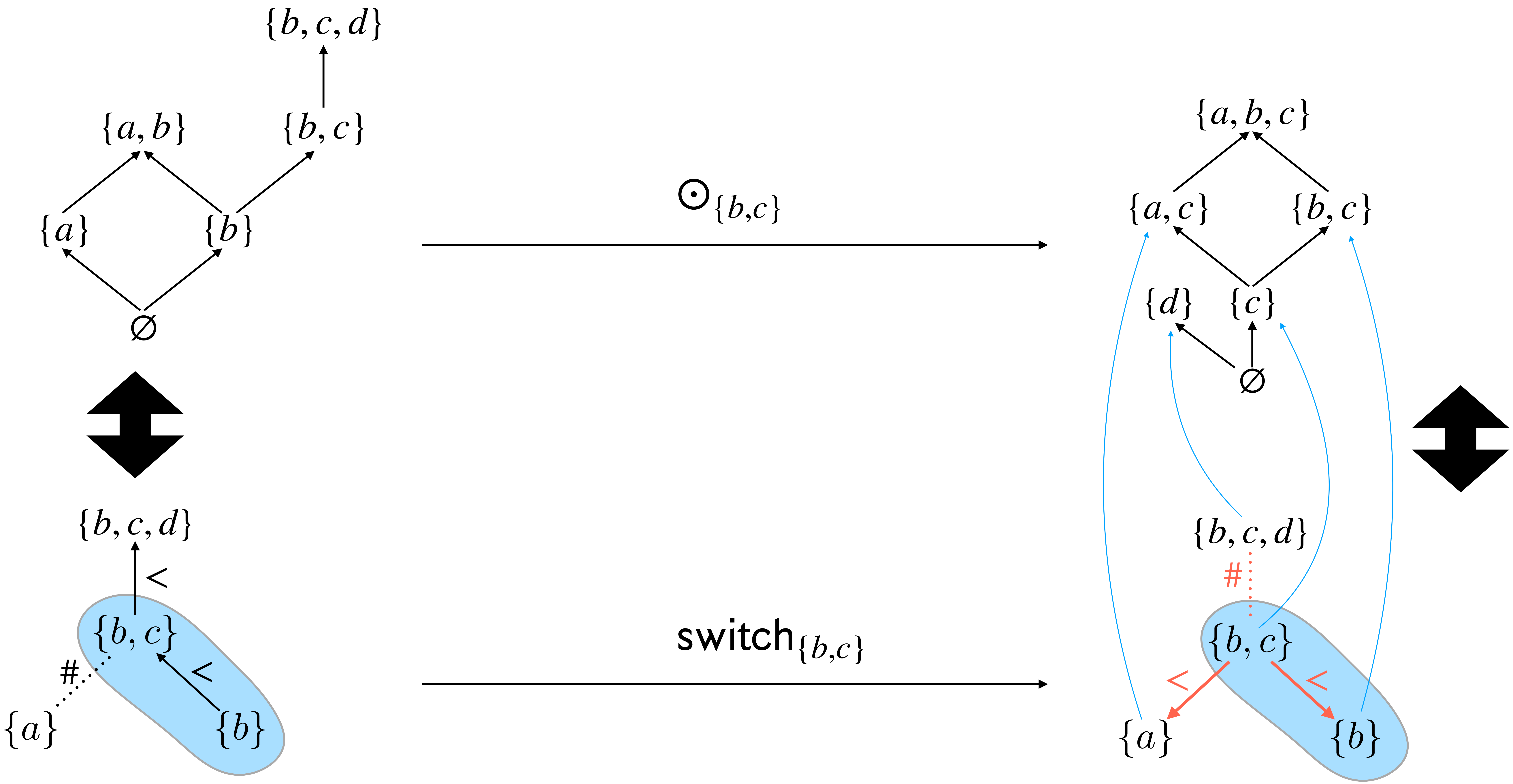
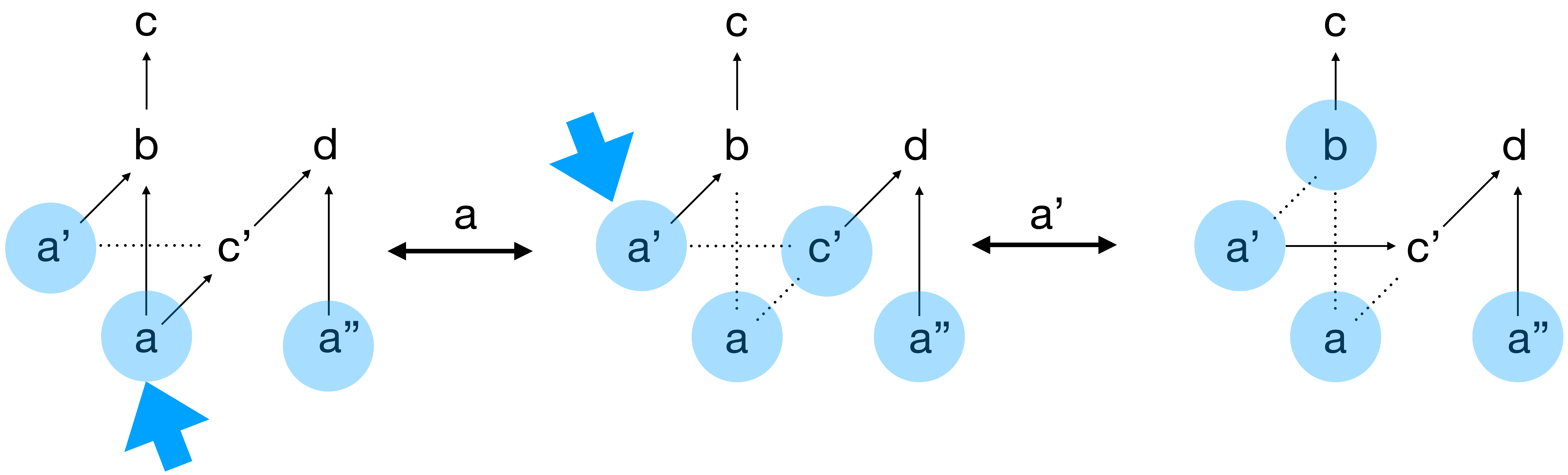


Event structure = local (reversible) computations



Take away

Any concurrent formalism with stable (prime algebraic coherent) causal structure (CCS, CSP, Pi-calculus etc.) …

Can be equipped with a (causally consistent) reversible semantics by making sure transition steps implement a switch.

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$$\sum_i a_i \, . \, P_i \to_{a_j} P_j \quad \text{(sum)} \qquad\qquad \frac{P \to_a P'}{P \| Q \to_a P' \| Q} \quad \text{(par)}$$

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Can be equipped with a (causally consistent) reversible semantics by making sure transition steps implement a switch.

$$\sum_i a_i \,.\, P_i \rightarrow_{a_j} P_j + a_j \,.\, \sum_{i \neq j} a_i \,.\, P_j \quad \text{(rsum)}$$